Auctions with Heterogeneous Entry Costs*

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Abstract

It is well known that if bidders have independent private values and homogeneous entry costs a first- or second-price auction with a reserve price equal to the seller's value maximizes social surplus and seller revenue, and leaves bidders with no surplus. We show that if entry costs are heterogeneous and private information, then the revenue maximizing reserve price is above the seller's value, a positive admission fee (and a reserve price equal to the seller's value) generates more revenue, and an entry cap combined with an admission fee generates even more revenue. In each case bidders capture informational rents. Nevertheless, social surplus and seller revenue coincide asymptotically, and are the same whether entry costs are homogeneous or heterogeneous, even though the effect of an increase in the number of bidders may differ. Our results are framed in terms of screening values rather than reserve prices, and apply to any standard auction.

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1 Introduction

A classic result of the auction literature is that in a standard auction with an exogenously fixed number of bidders who have independent private values, maximizing seller revenue requires screening bidders; i.e., the rules of the revenue maximizing auction are such that a bidder whose value is below the *screening value* will find it unprofitable to bid. Moreover, the revenue maximizing screening value is above the seller's value and is independent of the number of bidders – see Myerson (1981) and Riley and Samuelson (1981). In first- and second-price sealed-bid auctions, for example, the screening value is just the reserve price. Hence the revenue maximizing reserve price is above the seller's value and is independent of the number of bidders.

In many instances, however, the number of bidders is endogenously determined as the result of costly entry decisions. As noted by Milgrom (2004), "... auctions for valuable yet highly specialized assets often fail because of insufficient interest by bidders ... [since] buyers are naturally reluctant to begin an expensive, time-consuming evaluation of an asset when they believe that they are unlikely to win at a favorable price." Indeed, McAfee and McMillan (1987) and Levin and Smith (1994) have shown that endogenous entry has important implications in first- and second-price sealed-bid auctions. Specifically, when all buyers have the same (homogeneous) entry cost, a reserve price equal to the seller's value is optimal both for the seller and for society. Henceforth we use the term *buyer* to refer to an agent potentially interested in buying the object, and the term *bidder* to refer to a buyer who has entered the auction.

We study standard auctions with endogenous entry, but where buyers have heterogenous privately known entry costs. In the sale of a firm, for example, buyers may face different regulatory restrictions: some buyers may have to seek approval by regulatory authorities while others may not. Hence different buyers may have substantially different costs of discovering their value for the firm. Another example is Internet auctions, where a buyer's cost of discovering her value is the opportunity cost of her time, and it varies across buyers.

In our setting, like in McAfee and McMillan (1987) and Levin and Smith (1994), buyers simultaneously choose whether to enter the auction. Each buyer who enters the auction observes her value for the object and then bids. Our setting differs in that each buyer's entry cost is an independent draw from a common distribution, and is privately observed prior to entry. Our theoretical analysis provides a richer framework for empirical studies of auctions using data either from the field or from experiments – see, e.g., Li and Zheng (2009), Reiley (2006).

Heterogeneity of entry costs leads to results substantially different from those obtained when entry costs are homogenous. We show that while a screening value equal to the seller's value remains socially optimal, the revenue maximizing screening value is above the seller's value. (Thus, in first- and second-price sealed-bid auctions, for example, the revenue maximizing reserve price is above the seller's value.) Nevertheless, it is always below the revenue maximizing screening value when the number of bidders is exogenously fixed. Moreover, the revenue maximizing screening value depends on the number of buyers as well as on the distribution of values and entry costs.

When entry costs are homogenous, the seller has no incentive to charge an admission fee or subsidy (i.e., a fee which a buyer must pay, in addition to her entry cost, in order to learn her value).[1] We show that when entry costs are heterogeneous, if an admission fee is feasible, then the revenue maximizing screening value is, once again, the seller's value, and the revenue maximizing admission fee is positive. In other words, if it is feasible to screen buyers by entry costs, then it is suboptimal to screen bidders by values.

Paradoxically, although the seller always benefits, *ceteris paribus,* from an additional bidder in the auction, we show that it is in his interest to limit entry via a cap on the number of entrants. The seller obtains more revenue with an entry cap and an admission fee than he obtains with an admission fee and/or a screening value alone, whether entry costs are homogeneous or heterogenous.[2]

Our next set of results concerns the comparative static and asymptotic properties of equilibrium. For homogeneous entry costs, Levin and Smith (1994) show that

[1] In the literature, "entry fee" usually refers to a fee paid by the bidder to submit a bid when she already knows her value. Such a fee is captured in our setting through its effect on the screening value. An admission fee is paid by buyers *before* learning their values, and does not affect the result of the auction for a given number of bidders.

[2] Assuming, when entry cost are homogeneous, that bidders enter according to the mixed strategy entry equilibrium.

seller revenue decreases with the number of buyers in an entry equilibrium in mixed strategies. We describe simple examples that show that this result does not hold when entry costs are heterogeneous: an increase in the number of buyers may either increase or decrease seller revenue depending upon the distribution of values and entry costs. As the number of buyers grows large, auctions with homogenous and heterogeneous entry costs are closely related. We show that when the screening value and admission fee are both zero, then seller revenue is asymptotically the same when (i) buyers have a homogenous entry cost $c > 0$, and (ii) when buyers have heterogenous entry costs and the lower bound of entry costs is $\underline{c} = c$. Hence heterogeneity of entry costs does not matter asymptotically. Moreover, asymptotic seller revenue equals the constrained maximum social surplus (i.e., the maximum social surplus that can be obtained when all buyers enter independently and with the *same* probability). Thus, seller revenue is asymptotically the same whether the screening value and the admission fee are both set to zero or whether they are set to maximize seller revenue.

An entry cap, in contrast, remains advantageous for the seller even as the number of buyers grows large. When entry costs are homogeneous, the seller captures the entire *unconstrained* maximum social surplus by capping entry at the number of bidders that maximizes social surplus and simultaneously setting an admission fee which makes buyers indifferent between applying or not applying for entry. When entry costs are heterogenous and the lower bound $\underline{c}$ of entry costs is positive, then the seller asymptotically captures the unconstrained maximum social surplus by capping entry at the number of bidders that would maximize surplus if all buyers had the same entry cost $\underline{c}$ and employing an admission fee. When the lower bound of entry costs is zero and bidders' values are distributed uniformly, there is asymptotically no advantage to employing an entry cap: seller revenue is asymptotically the unconstrained maximum social surplus without screening buyers by entry costs or by values, and without capping the number of entrants.

In order to understand the intuition for our results, it is useful to review the results and intuition when entry costs are homogeneous. Let us assume for simplicity that the seller's value for the object is zero. A key result in this setting is that in a standard auction with a screening value of zero the contribution to social surplus of an additional bidder is exactly equal to the buyer's utility to entering.[3] Thus,

[3] A version of this result is established in Engelbrecht-Wiggans (1993)'s Proposition 1, and is also

when entry costs are homogeneous, the interests of an entrant and of society are aligned: a buyer enters only if her expected utility to entering is above her entry cost; that is, only if her contribution to social surplus is positive. Hence the number of entering buyers maximizes social surplus. If the auction is sufficiently competitive, then in equilibrium each buyer is indifferent between entering or not. Therefore buyer surplus is competed away and the seller captures the entire social surplus. Hence a screening value equal to zero maximizes both seller revenue and social surplus.

When entry costs are heterogeneous a version of the key result described above also holds: we show that in a standard auction with a screening value of zero the contribution to social surplus of a marginal increase of the entry threshold is proportional to a buyer's utility to entering; that is, the interests of buyers and society are also aligned when entry costs are heterogenous. Consequently, a standard auction with a zero screening value maximizes social surplus whether entry costs are homogeneous or heterogeneous. With heterogeneous entry costs, however, not all buyer surplus is competed away by entry: whereas the surplus of a buyer with an entry cost equal to the equilibrium threshold is exactly zero, the surplus of buyers with lower entry costs (who also enter) is positive. Therefore buyers capture a positive share of the surplus. And even though setting a positive screening value reduces social surplus (because it reduces entry below the socially optimal level and also leads to ex-post inefficiencies), it increases the seller's share of social surplus and, as we show, increases revenue.

If an admission fee is feasible, an even greater revenue can be obtained with a positive admission fee and a screening value equal to the seller's value (i.e., zero): reducing the screening value to zero and introducing an admission fee that leaves unchanged the utility to a buyer to entering the auction induces the same entry by buyers without incurring the ex-post inefficiencies of a positive screening value. Thus, seller revenue increases since social surplus increases while total buyer surplus is unchanged.

Since entry decisions are independent, with positive probability either too many or too few buyers enter the auction. Thus, there is a trade-off between competition and surplus creation, which is not solved by setting a reserve price or admission fee. When entry costs are homogeneous and buyers enter according to the symmetric

observed in both McAfee and McMillan (1987) and Levin and Smith (1994).

mixed strategy equilibrium, this trade-off is most obvious as social surplus falls as there are more buyers. In this case, an appropriate entry cap and revenue maximizing admission fee solve the problem, allowing the seller to capture the unconstrained maximum social surplus.

When entry costs are heterogenous, the trade-off between competition and surplus creation remains, even though social surplus may rise or fall as the number of buyers grows. An appropriate entry cap reduces excessive entry and, *ceteris paribus* (i.e., holding entry decisions fixed), raises social surplus. This entry cap combined with a revenue maximizing admission fee raises social surplus, reduces total buyer surplus, and hence raises seller revenue.

Related Literature

In our setting, buyers make entry decisions *before* they observe their values, and entry costs (interpreted as valuation-discovery costs) are heterogeneous and private information. Samuelson (1985) studies a procurement sealed-bid auction with entry where buyers make entry decisions *after* observing their "values" (i.e., their procurement costs), and entry costs (interpreted as bid-preparation costs) are homogenous. Samuelson (1985) shows that if the reserve is equal to the bidder's value, then equilibrium is socially optimal. In this setting, Menezes and Monteiro (2000) study the equilibria of first- and second-price sealed-bid auctions, and provide an interesting characterization of the optimal auction. Tan and Yilankaya (2007) study second-price auctions and provide conditions under which the entry equilibrium is unique (and symmetric), and under which there are other (asymmetric) equilibria.

In Samuelson's setting, both reserve prices and entry fees screen bidders by values, and are thus interchangeable. In our model, by contrast, reserve prices (and/or entry fees) screen bidders by values, while admission fees screen buyers by entry costs. We show that when both instruments are available, maximizing seller revenue entails screening buyers by entry costs (by setting a positive admission fee), but not by values (i.e., the revenue maximizing screening value is the seller's value).

Green and Laffont (1984) study the existence of equilibrium in a model where, as in our setting, both entry costs and values are private information, but they assume, as in Samuelson (1985), that a buyer makes entry decisions having observed both her entry cost and her value. Kaplan and Sela (2003) study auctions where entry costs are private information, but bidders' values are commonly known. Lu (2010) provides an interesting characterization of the revenue maximizing admission fees in second price sealed-bid auctions with heterogenous entry costs. Pevnitskaya (2004) studies endogenous entry in first-price sealed-bid auctions with heterogeneous risk attitudes.

The paper is organized as follows. In Section 2 we lay out the basic setting. Section 3 reviews the results for homogenous entry costs. Section 4 presents our results for heterogenous entry costs. Section 5 develops a numerical example comparing screening values, admissions fees, and entry caps. Section 6 studies the effect of increasing the number of buyers. Section 7 concludes. Proofs are in the Appendix.

2 Preliminaries

Consider a market for a single object for which there are N risk-neutral buyers and a risk-neutral seller. In this market the object is allocated using a *standard* auction (i.e., an anonymous auction that allocates the object to the highest bidder) with a *screening value* $v \in [0, \bar{v}]$. Each buyer must decide whether to enter the auction thereby incurring an entry cost. A buyer who enters the auction learns her value, and becomes a bidder. Buyers' values $V_1, \ldots, V_N$ are independently and identically distributed on $[0, \bar{v}]$ according to an increasing *c.d.f.* F with an increasing hazard rate, and *p.d.f.* f. The seller's value for the object is zero.

The screening value v is the minimum value for which bidding is worthwhile; i.e., the lowest bidder type that bids. The screening value captures everything about the rules of a standard auction that is payoff-relevant (e.g., the payment rule, the reserve price, the entry fee, etc.). The impact on the entry game of any change in these rules can be captured as a change of the screening value.

Auctions with a Fixed Number of bidders

By the Revenue Equivalence Theorem – Myerson (1981), Riley and Samuelson (1981) – in an increasing symmetric equilibrium of a standard auction with $n \geq 1$

bidders, the revenue of the seller is[4]

$$\pi(v,n) = n\int_v^{\bar{v}} (yf(y) + F(y) - 1)F^{n-1}(y)dy,$$

the utility of a bidder is

$$u(v,n) = \int_v^{\bar{v}} \left(\int_v^y F(x)^{n-1}dx\right) f(y)dy,$$

and the social surplus is

$$s(v,n) = \int_v^{\bar{v}} ydF^n(y).$$

We note that $\pi(v,n)$ is increasing in n, $u(v,n)$ is decreasing in both v and n, and $s(v,n)$ is decreasing in v and increasing in n. Also, it is easy to show that

$$s(v,n) = \pi(v,n) + nu(v,n).$$

Denote by $V_{(n)}$ the highest order statistic of $\{V_1,\ldots,V_n\}$. Then

$$s(0,n) = E(V_{(n)}),$$

i.e., a standard auction with a screening value equal to zero realizes the maximum surplus.

Proposition 1 below establishes that when the screening value is zero, the utility of each bidder is equal to her contribution to social surplus. We provide a simple proof of this result in the Appendix. Proposition 1 of Engelbrecht-Wiggans (1993) establishes a version of this formula for second-price auctions.

Proposition 1. *In a standard auction with a screening value of zero, the utility of a bidder is her contribution to social surplus, i.e.,* $u(0,1) = s(0,1)$ *and* $u(0,n) = s(0,n) - s(0,n-1)$ *for* $n > 1$.

As will be seen later, this fact is key to understanding the intuition for the results on entry with homogeneous entry costs.

[4]For brevity of exposition, throughout the paper we omit the term *expected* when referring to expected seller revenue, expected social surplus, etc.

The Entry Game

Assume that each buyer enters the auction with probability p. Then the number of bidders follows a binomial distribution $B(N, p)$. Write $p_n^N(p)$ for the probability that the number of bidders is $n \in \{0, 1, \ldots, N\}$. Also assume that the screening value $v \in [0, \bar{v}]$ is independent of the number of bidders n. Then seller revenue is

$$\Pi(v, p) = \sum_{n=1}^{N} p_n^N(p)\pi(v, n),$$

the utility of a buyer to entering the auction is

$$U(v, p) = \sum_{n=0}^{N-1} p_n^{N-1}(p)u(v, n + 1),$$

and the gross social surplus is

$$S(v, p) = \sum_{n=1}^{N} p_n^N(p)s(v, n).$$

Since $s(v, n) = \pi(v, n) + nu(v, n)$, then

$$S(v, p) = \Pi(v, p) + NpU(v, p). \tag{1}$$

It is easy to see that $U(v, p)$ is decreasing in p: if $p'' > p'$, then $B(N, p'')$ first order stochastically dominates $B(N, p')$, and therefore since $u(v, n)$ is decreasing in n, we have $U(v, p'') < U(v, p')$. Also, since $u(v, n)$ is decreasing in v, then $U(v, p)$ is also decreasing in v.

We study the symmetric equilibria of the *entry game*. In this game, the payoff to a buyer who enters, when every other buyer enters with the same probability p, is $U(v, p)$ minus her entry costs.

Our assumption that the screening value is independent of the number of bidders n is appropriate when either (i) the rules of the auction are such that the screening value is the same for every n, or (ii) bidders do not observe the number of bidders present in the auction so that their bidding strategies are independent of n.[5] The

[5] The Revenue Equivalence Theorem applies even when there is uncertainty about the number of bidders in the auction, provided that bidders have symmetric expectations – see Krishna (2002), Section 3.2.2.

former holds in first, second, and k^{th} price sealed-bid auctions, for example, where the screening value equals the reserve price regardless of the number of bidders. In this case whether bidders observe the number of entrants is irrelevant (i.e., their payoffs in the entry game are the same). In contrast, in an all-pay auction with a fixed reserve price, the formulas above describe the payoffs in the entry game only if bidders do not observe the number of entrants.

3 Homogenous entry costs

In this section we derive existing results identifying the revenue maximizing screening value when all buyers have the same fixed entry cost $c > 0$, and show that these results hold for any standard auction. We assume that $u(0, N) < c < u(0, 1)$ to rule out uninteresting equilibria in which either every buyer or no buyer enters.

If n buyers enter the auction, the maximum social surplus that can be realized is

$$E(V_{(n)}) - nc = s(0, n) - nc.$$

Since $u(0, n) = s(0, n) - s(0, n-1)$ by Proposition 1, then the contribution to social surplus of the n-th buyer to enter is

$$s(0, n) - s(0, n-1) - c = u(0, n) - c.$$

Since $u(0, n)$ is decreasing in n, this contribution is decreasing in n.

Consider a standard auction with a zero screening value. In a pure strategy equilibrium of the entry game, the n-th buyer enters if her payoff to entering, $u(0, n)$, is above her cost, c, and does not enter if it is below; i.e., a buyer enters if and only if her entry raises social surplus. Therefore the number of entering buyers n^* maximizes social surplus. If we ignore that n^* must be an integer, then buyers capture none of the surplus (i.e., $u(0, n^*) - c = 0$), and the seller captures the entire social surplus. A positive screening value reduces the social surplus and, because seller revenue is at most the social surplus, also reduces seller revenue. Hence the revenue maximizing screening value is zero.[6]

[6]Since the number of entrants is an integer, however, bidder surplus will typically be positive, and may be non-negligible. We address this issue in Proposition 7.

The key insight above was that the private and social benefit of entry coincide in a standard auction with a screening value equal to zero. Levin and Smith (1994) show that the same logic applies to symmetric entry equilibria in mixed strategies. If each buyer enters with probability p, then the number of bidders follows a binomial distribution $B(N,p)$, and the maximum social surplus that can be achieved is

$$\sum_{n=1}^{N} p_n^N(p)E(V_{(n)}) - Npc = S(0,p) - Npc. \tag{2}$$

A standard auction with a screening value equal to zero attains this maximum. Note that this is a *constrained* maximum surplus; i.e., it is the maximum surplus when all buyers enter with the *same* probability. Using Proposition 1 we can calculate

$$\begin{aligned}\frac{dS(0,p)}{dp} &= N\left(\sum_{n=1}^{N} p_{n-1}^{N-1}(p)s(0,n) - \sum_{n=1}^{N-1} p_n^{N-1}(p)s(0,n)\right)\\ &= N\sum_{n=0}^{N-1} p_n^{N-1}(p)u(0,n+1)\\ &= NU(0,p),\end{aligned}$$

i.e., the marginal contribution to gross social surplus of an increase in the probability of entry is proportional to the utility of an entering buyer. Since U is decreasing in p, then

$$\frac{d^2S(0,p)}{dp^2} = N\frac{dU(0,p)}{dp} < 0.$$

Hence the social surplus, $S(0,p) - Npc$, is a concave function of p whose maximum on $[0,1]$ is attained at the solution to the equation

$$N(U(0,p) - c) = 0.$$

In the symmetric mixed strategy entry equilibrium, p^*, buyers are indifferent between entering or not, i.e., $U(0,p^*) - c = 0$. Therefore the social surplus is maximized.[7] Since the seller captures the entire social surplus, the revenue maximizing screening value is zero.

[7] The assumption $u(0,N) < c < u(0,1)$ implies that $1 < n^* < N$, and that the unique symmetric entry equilibrium p^* satisfies $p^* \in (0,1)$. The social surplus when bidders enter with probability p^* is less than when exactly n^* bidders enter, since with positive probability either too many or too few bidders enter the auction. Thus, in the mixed strategy equilibrium the social surplus is *constrained* maximized.

These results are summarized in the proposition below.

Proposition MM-LS. *(Homogeneous entry costs – McAfee and McMillan (1987), Levin and Smith (1994).) In a standard auction with a screening value equal to zero, if buyers follow a (symmetric mixed) pure strategy entry equilibrium, then the (constrained) maximum social surplus is realized and is captured by the seller. Hence either a first- or a second-price sealed-bid auction with a reserve price equal to zero maximizes seller revenue.*

4 Heterogenous entry costs

In this section we study the general case where buyers have heterogenous entry costs. Specifically, each buyer i has a privately known entry cost Z_i. Buyers' entry costs $Z_1, \ldots, Z_N$ are independently and identically distributed according to a *c.d.f.* H with support $[\underline{c}, \bar{c}]$, where $0 < \underline{c} < \bar{c} \leq \infty$. As in the homogenous entry cost case (i.e., the case where H is degenerate), we assume that $u(0, N) < \bar{c}$ and $\underline{c} < u(0, 1)$ to rule out uninteresting equilibria. For simplicity, we assume also that H is increasing, satisfies $H(\underline{c}) = 0$, and has a *p.d.f.* h.

In this setting, an entry strategy for a buyer can be described by a threshold $t \in [\underline{c}, \bar{c}]$ indicating the maximum entry cost for which the buyer enters the auction; that is, a buyer enters when her entry cost is less than t, and does not enter if it is greater than t – whether a buyer enters when her entry cost is exactly t is inconsequential.[8] If all buyers employ the same threshold t, then the number of bidders follows a binomial distribution $B(N, H(t))$.

Consider any standard auction with a screening value $v \in [0, \bar{v}]$ and an *admission fee* (or subsidy) $\phi \in \mathbb{R}$ which a buyer must pay, in addition to her entry cost, in order to enter. When all buyers enter according to a common threshold t, then the payoff to a buyer with entry cost z who enters is $U(v, H(t)) - z - \phi$. A *symmetric entry equilibrium* is a threshold $t \in [\underline{c}, \bar{c}]$ such that for all $z \in [\underline{c}, \bar{c}]$: $U(v, H(t)) > z + \phi$ implies $t > z$, and $U(v, H(t)) < z + \phi$ implies $t < z$; i.e., a buyer enters if her utility

[8] In general, entry decisions are described by a mapping from $[\underline{c}, \bar{c}]$ into $[0, 1]$ indicating for each entry cost the probability with which a bidder enters the auction. When H is atomless, however, in equilibrium buyers follow a threshold strategy.

to entering exceeds the sum of her entry cost z and the admission fee ϕ, and does not enter if it is below.

As we shall see, when entry costs are heterogeneous, an admission fee, if feasible, is advantageous to the seller. We therefore introduce admission fees from the outset. For each screening value $v \in [0, \bar{v}]$ and admission fee $\phi \in \mathbb{R}$, denote by $t^*(v, \phi)$ the symmetric equilibrium threshold. Proposition 2 establishes that for every v and ϕ there is a unique symmetric entry equilibrium, i.e., $t^*(v, \phi)$ is a well defined function.[9]

Proposition 2. *For each screening value $v \in [0, \bar{v}]$ and admission fee $\phi \in \mathbb{R}$, there is a unique symmetric entry equilibrium $t^*(v, \phi) \in [\underline{c}, \bar{c}]$. The mapping t^* is a continuous function. When the equilibrium is interior, $t^*(v, \phi)$ solves*

$$U(v, H(t)) = t + \phi, \tag{3}$$

and is decreasing in both v and ϕ.

Given a common entry threshold $t \in [\underline{c}, \bar{c}]$, the social surplus generated in a standard auction with a screening value of v is

$$W(v, t) = S(v, H(t)) - Nc(t), \tag{4}$$

where

$$c(t) = \int_{\underline{c}}^{t} z dH(z)$$

is the expected entry cost incurred by each buyer. Write

$$W^* = \max_{(v,t) \in [0,\omega] \times [\underline{c}, \bar{c}]} W(v, t), \tag{5}$$

for the *constrained maximum social surplus*. W^* is a constrained maximum in the sense that buyers enter *independently* according to a *symmetric* entry rule.

Recall that a standard auction in which the screening value and admission fee are both equal to zero maximizes social surplus when entry costs are homogeneous. Proposition 3 establishes that this result holds as well when entry costs are heterogeneous. In particular, the symmetric entry equilibrium threshold $t^*(0, 0)$ induces socially optimal entry.

[9] Tan and Yilankaya (2006) obtain an analogous result in their framework.

Proposition 3. *A screening value and an admission fee both equal to zero maximize social surplus, i.e.,* $W(0, t^*(0,0)) = W^*$.

If the entry equilibrium is interior, then $U(v, H(t^*(v,\phi))) - \phi = t^*(v,\phi)$. Hence *total buyer surplus* is

$$N \int_{\underline{c}}^{t^*(v,\phi)} [U(v, H(t^*(v,\phi))) - \phi - z] dH(z) = N \int_{\underline{c}}^{t^*(v,\phi)} [t^*(v,\phi) - z] dH(z) > 0, \quad (6)$$

i.e., buyers have information rents. Thus, the seller does not capture the entire social surplus. By Proposition 2, t^* is decreasing in both v and ϕ, and hence total buyer surplus decreases with both v and ϕ. Proposition 4 summarizes these results.

Proposition 4. *In an interior entry equilibrium, total buyer surplus is positive and decreasing in both the screening value and the admission fee, and seller revenue is less than the social surplus.*

In the rest of this section we study revenue maximizing screening values, admission fees, and entry caps. Seller revenue is the sum of revenue from the auction, $\Pi(v, H(t^*(v,\phi)))$, and revenue from admission fees, $NH(t^*(v,\phi))\phi$. Using equation (1) evaluated at $p = H(t^*(v,\phi))$, the equilibrium condition (3), and equation (4) above, seller revenue can be written as

$$\Pi(v, H(t^*(v,\phi))) + NH(t^*(v,\phi))\phi = W(v, t^*(v,\phi)) - N \int_{\underline{c}}^{t^*(v,\phi)} [t^*(v,\phi) - z] dH(z). \quad (7)$$

This equation has a clear interpretation: seller revenue is simply the difference between the social surplus ("revenue") and total buyer surplus ("cost").

Screening Values

We begin by studying revenue maximizing screening values when admission fees are not feasible (i.e., assuming that $\phi = 0$). It is well known that if the number of bidders is exogenously given, then the revenue maximizing screening value v^F is positive and is the solution to the equation

$$v = \frac{1 - F(v)}{f(v)},$$

independently of the number of bidders – see Myerson (1981) and Riley and Samuelson (1981). Recall that when entry is endogenous and costs are homogeneous, the revenue maximizing screening value is zero. Proposition 5 establishes that when entry costs are heterogeneous, a revenue maximizing screening value is between these two values, i.e., $v^* \in (0, v^F)$, and optimally trades off "revenue" and "cost" effects.

Proposition 5. *A revenue maximizing screening value* v^* *exists, satisfies* $0 < v^* < v^F$, *and is characterized by the equation*

$$\frac{\partial W}{\partial v} + \frac{\partial W}{\partial t}\frac{\partial t^*}{\partial v} = NH(t^*(v,0))\frac{\partial t^*}{\partial v}. \tag{8}$$

The intuition for why a revenue maximizing screening value is positive is as follows: when the screening value is zero, a marginal increase in the screening value has a negative impact on both social surplus and total buyer surplus. Since social surplus is maximized when the screening value is zero (Proposition 3), the impact on social surplus is negligible. The impact on total buyer surplus, however, is non-negligible (see Lemma 2). Hence seller revenue, which is social surplus less total buyer surplus, increases.

A similar argument shows that a revenue maximizing screening value is below v^F: a marginal decrease in the screening value from v^F has a negative (direct) impact on revenue holding the entry threshold $t^*(v^F, 0)$ fixed, and a positive (indirect) impact on revenue through increased entry. Since for a fixed entry threshold seller revenue is maximized at v^F, i.e., $\frac{\partial \Pi(v,p)}{\partial v}|_{v=v^F} = 0$, the first effect is negligible. However, the effect on revenue of increasing the entry threshold is non-negligible (see Lemma 4).

Equation (8) shows the trade-offs facing the seller: changing the screening value has an impact on both social surplus, a *revenue* effect, and total buyer surplus, a *cost* effect. The revenue maximizing screening value balances these two effects, equating *marginal revenue* and *marginal cost*. The solution to equation (8) depends on all the primitives: the distributions of values and entry costs (F and H), and the number of buyers (N). In contrast, when all buyers have the same entry cost c, the revenue maximizing screening value is zero independently of F, N, and c. And when entry is exogenous, the revenue maximizing screening value depends on F but is independent of N.

Admission Fees

Assume now that the seller may set an admission fee ϕ as well as a screening value v. While a buyer's entry cost represents her own idiosyncratic cost of discovering her value, the admission fee is an extra cost that the seller imposes on a buyer who chooses to enter the auction. A buyer might, for example, need to view the item for auction in order to discover her value, in which case the seller may charge the buyer for making the item available.

Proposition 6 establishes that an admission fee enables the seller to obtain more revenue than he obtains by choosing a screening value alone. Indeed, when an admission fee is feasible, then the revenue maximizing admission fee is positive and the revenue maximizing screening value is zero; i.e., it is optimal to screen buyers by entry costs, but it is suboptimal to screen bidders by values. Proposition 6 characterizes the revenue maximizing admission fee.

Proposition 6. *If an admission fee is feasible, then the revenue maximizing screening value is zero, i.e., if it is feasible to screen buyers by entry costs, then it is suboptimal to screen bidders by values. Further, a revenue maximizing admission fee ϕ^* exists, is positive, and is characterized by the equation*

$$\frac{\partial W}{\partial t}\frac{\partial t^*}{\partial \phi} = NH(t^*(0,\phi))\frac{\partial t^*}{\partial \phi}. \tag{9}$$

Moreover, seller revenue is greater than when an admission fee is not feasible.

It is easy to see that the revenue maximizing screening value is zero when an admission fee is feasible: if the screening value is positive, then the seller can reduce the screening value to zero and at the same time raise the admission fee so that the utility to a buyer to entering the auction is unchanged. This admission fee (combined with a zero screening value) induces the same entry by buyers without incurring the ex-post inefficiencies of a positive screening value. Seller revenue must increase since social surplus increases while total buyer surplus is unchanged.

Clearly, a negative admission fee is suboptimal since raising the fee to zero increases social surplus (by Proposition 3) and decreases total buyer surplus (by Proposition 4), thereby increasing seller revenue. An admission fee of zero is also suboptimal: increasing the admission fee above zero reduces both social surplus and total buyer surplus; the effect on social surplus is negligible since $\partial W(0, t^*(0,0))/\partial t = 0$ (Proposition 3), while the effect on total buyer surplus is not non-negligible since $NH(t^*(0,0))\partial t^*/\partial \phi < 0$; i.e., seller revenue increases with ϕ near zero. (A revenue maximizing admission fee balances these two effects as equation (9) requires.) Therefore a revenue maximizing admission fee is positive and induces less entry than socially optimal.

Unlike a screening value, an admission fee only has an indirect effect on the social surplus since it affects entry decisions, but does not alter the social surplus generated in the auction taking as given the number of bidders.

ENTRY CAPS

We examine now the consequences of introducing an *entry cap* $\bar{n} < N$; i.e., a cap on the number of bidders. In this new scenario, a buyer must decide whether to apply for entry. Applying for entry entails a commitment to enter the auction and pay the admission fee *if admitted*. When $\bar{n}$ or fewer buyers apply for entry, then each applicant is admitted. When more than $\bar{n}$ buyers apply, then applicants are anonymously (i.e., symmetrically) rationed so that exactly $\bar{n}$ are admitted; hence every buyer who applies has the same probability of being admitted. Since the revenue maximizing screening value is zero when an admission fee is feasible (Proposition 6), we assume the seller employs an admission fee, but sets the screening value to zero.

When entry costs are homogeneous, an entry cap combined with an admission fee allows the seller to capture the entire unconstrained maximum social surplus. Assume that all buyers have the same entry cost $c > 0$. Recall that n^* is the number buyers that maximizes social surplus, i.e., n^* is the largest integer such that $u(0, n^*) - c \geq 0$. In an auction with an entry cap $\bar{n} = n^*$ and an admission fee $\phi = u(0, \bar{n}) - c$ the payoff to a buyer who is admitted if $n < \bar{n}$ buyers apply is

$$u(0, n) - c - \phi > u(0, \bar{n}) - c - \phi = 0,$$

and it is zero if $\bar{n}$ or more buyers apply. Hence applying is a weakly dominant strategy. Further, in equilibrium $\bar{n}$ or more buyers apply, and in a symmetric equilibrium every buyer applies. Therefore in equilibrium the number of bidders is $\bar{n}$, total buyer surplus is zero, and the unconstrained maximum social surplus is realized and captured by the seller. Moreover, varying the number of buyers N does not affect either social surplus or seller revenue, so long as $N > n^*$.[10] These results are summarized in Proposition 7.

Proposition 7. *Assume that all buyers have the same entry cost $c > 0$. Then an entry cap $\bar{n} = n^*$, and an admission fee $\phi = u(0, \bar{n}) - c$ (and a screening value of zero) maximizes seller revenue and social surplus. Moreover, the seller captures the unconstrained maximum social surplus. An increase in the number of buyers N has no effect on either social surplus or seller revenue.*

Thus, the entry cap $\bar{n} = n^*$ rules out the possibility that there are too many or too few bidders, as occurs in the symmetric mixed strategy entry equilibrium identified by Levin and Smith (1994), and the admission fee $\phi = u(0, \bar{n}) - c$ eliminates the rents that may be captured by buyers in the pure strategy equilibria identified by McAfee and McMillan (1987).

When entry costs are heterogenous, a buyer's decision whether to apply for admission depends on her entry cost. Let $\bar{n} \in \{1, \ldots, N-1\}$ be a binding entry cap. When each buyer applies for admission with probability p, then a buyer's utility conditional on being admitted is

$$\bar{U}(p) = \sum_{n=0}^{\bar{n}-1} \frac{p_n^{N-1}(p)}{\alpha(p)} u(0, n+1) + \sum_{n=\bar{n}}^{N-1} \frac{p_n^{N-1}(p)}{\alpha(p)} \frac{\bar{n}}{n+1} u(0, \bar{n}),$$

where

$$\alpha(p) = \sum_{n=0}^{\bar{n}-1} p_n^{N-1}(p) + \sum_{n=\bar{n}}^{N-1} p_n^{N-1}(p) \frac{\bar{n}}{n+1}$$

is the probability that a buyer who applies is admitted.[11] Note that $\alpha(0) = 1$ and $0 < \alpha(p) < 1$ for $p > 0$.

A *symmetric equilibrium* is a threshold $\bar{t} \in [\underline{c}, \bar{c}]$ such that for all $z \in [\underline{c}, \bar{c}]$: $\bar{U}(H(\bar{t})) > z + \phi$ implies $\bar{t} > z$, and $\bar{U}(H(\bar{t})) < z + \phi$ implies $\bar{t} < z$; i.e., a buyer applies for admission if her utility conditional on being admitted exceeds the sum of her entry cost and the admission fee, and does not apply otherwise.

[10] Without an entry cap, both social surplus and seller revenue decrease with N in the symmetric mixed strategy entry equilibrium – see Levin and Smith (1994), Proposition 9.

[11] For $n < \bar{n}$, the ratio $p_n^{N-1}(p)/\alpha(p)$ is the probability a bidder assigns to the event that n of the $N-1$ other bidders are admitted when she herself is admitted.

Denote by $n^*(c)$ the largest integer n such that $u(0, n) - c \geq 0$. Proposition 8 establishes that an entry cap raises seller revenue.

Proposition 8. *Assume that* $N > n^*(\underline{c})$. *Then an entry cap* $\bar{n} = n^*(\underline{c})$ *combined with a revenue maximizing admission fee and a zero screening value generates more revenue than any admission fee and/or screening value alone.*

The intuition for this result is as follows: suppose in the absence of an entry cap the seller sets a revenue maximizing admission fee ϕ^* and screening value $v = 0$ (see Proposition 6). Let $t^*(0, \phi^*)$ denote the equilibrium entry threshold. If the seller introduces an entry cap $\bar{n} = n^*(\underline{c})$, then a buyer whose entry cost is z and who is not admitted to the auction (as a result of more than $n > \bar{n}$ buyers applying) obtains a payoff of zero and makes a social contribution of zero. Had she been admitted, her contribution to social surplus, $u(0, n) - z$, would have been negative, since

$$u(0, n) - z \leq u(0, \bar{n} + 1) - \underline{c} < 0.$$

Also, the buyer is better off as a result of being excluded since her entry cost z exceeds her utility, $u(0, n)$, if admitted to an auction with $n > \bar{n}$ bidders. Hence, *ceteris paribus* (i.e., if buyers apply to the auction according to the threshold $t^*(0, \phi^*)$), both total buyer surplus and social surplus increase as a result of the entry cap.

Proposition 8 shows that if, in addition to the entry cap, the admission fee is raised (from ϕ^*) until the equilibrium threshold for *applying* for entry equals $t^*(0, \phi^*)$, then total buyer surplus decreases below its level without the entry cap. Thus, the introduction of the entry cap $\bar{n} = n^*(\underline{c})$, combined with an increase of the admission fee that leaves the threshold $t^*(0, \phi^*)$ unchanged, increases social surplus and decreases total buyer surplus, thereby leading to an increase in seller revenue.

5 An Example

Assume that $N = 2$, and that values and entry costs are distributed uniformly with $\bar{v} = 1$, $\underline{c} = 1/4$ and $\bar{c} = 1/2$. We calculate the equilibrium outcomes for a standard auction in four scenarios. In scenario (i) both the screening value and the admission fee are zero. In scenario (ii) the screening value is set to maximize revenue assuming that no admission fee is feasible. In scenario (iii) both the screening value and admission fee are set to maximize revenue. In scenario (iv) there is an entry cap and a revenue maximizing screening value and admission fee.

By Proposition 2, in scenarios (i)-(iii) the equilibrium threshold t solves equation (3), which in this example is

$$(1 - H(t))\, u(v,1) + H(t)u(v,2) = t + \phi,$$

where $H(t) = 4t - 1$, $u(v,1) = (1-v)^2/2$ and $u(v,2) = (2v+1)(1-v)^2/6$. Solving for t yields

$$t^*(v,\phi) = \frac{(5-2v)(1-v)^2 - 6\phi}{8(1-v)^3 + 6}.$$

Seller revenue is $\Pi(v, H(t^*(v,\phi))) + NH(t^*(v,\phi))\phi$, which becomes

$$2(1 - H(t^*(v,\phi)))H(t^*(v,\phi))\pi(v,1) + H(t^*(v,\phi))^2\pi(v,2) + 2H(t^*(v,\phi))\phi,$$

where $\pi(v,1) = v(1-v)$ and $\pi(v,2) = (1-v)(4v^2+v+1)/3$. Total buyer surplus is $N[H(t^*(v,\phi))t^*(v,\phi) - c(t^*(v,\phi))]$, which becomes

$$2\left(H(t^*(v,\phi))t^*(v,\phi) - \int_{1/4}^{t^*(v,\phi)} 4z dz\right).$$

We use these formulae to calculate the equilibrium in each scenario.

In scenario (i) we have $v = \phi = 0$. In order to calculate the revenue maximizing screening value of scenario (ii), we set $\phi = 0$ and solve $d\Pi(v, H(t^*(v,0)))/dv = 0$ to obtain $v^* = 0.0972$. In scenario (iii), by Proposition 6 the revenue maximizing screening value is $v = 0$ and the revenue maximizing admission fee solves

$$\frac{d}{d\phi}[\Pi(0, H(t^*(0,\phi))) + NH(t^*(0,\phi))\phi] = 0,$$

which yields $\phi^* = 0.075$. Applying the values of v and ϕ for scenarios (i)-(iii) to the formulae above, we calculate the equilibrium threshold, seller revenue, total buyer surplus, and social surplus. The numerical results are given in Table 1 below.

Scenario (iv) requires a separate analysis. By Proposition 8 we set $\bar{n} = n^*(\underline{c}) = 1$ and $v = 0$.[12] The equilibrium threshold $\bar{t}$ for *applying* to the auction solves

$$u(0,1) = \bar{t} + \phi.$$

[12] Recall that $n^*(\underline{c})$ is the largest integer n such that $u(0,n) \geq \underline{c}$. Since $u(0,1) = 1/2 > \underline{c} = 1/4 > u(0,2) = 1/6$, then $n^*(\underline{c}) = 1$.

Solving for $\bar{t}$ yields $\bar{t}^*(\phi) = \frac{1}{2} - \phi$. Since there is at most one bidder and the screening value is zero, the auction generates no revenue. Thus seller revenue is ϕ when at least one buyer applies and is zero otherwise; i.e., seller revenue is $[1 - (1 - H(\bar{t}^*(\phi)))^2]\phi$. The revenue maximizing admission fee is $\bar{\phi}^* = 0.1443$.

Table 1 describes the equilibrium outcomes in scenarios (i) to (iv). The values in parentheses in the last three columns are percentages of the baseline scenario (i) values. In scenario (ii), where no admission fee is feasible, a revenue maximizing screening value increases seller revenue by 18%, while total buyer surplus and social surplus decrease by nearly 45% and 9%, respectively. If an admission fee is feasible – scenario (iii) – then seller revenue increases by 22%, while total buyer surplus and social surplus decrease by 51% and 9%, respectively. An entry cap together with a revenue maximizing admission fee – scenario (iv) – increases seller revenue by 57%, decreases total buyer surplus by 24%, and *increases* social surplus by 22%. Social surplus exceeds the constrained maximum social surplus (i.e., the social surplus in scenario (i)), because buyers no longer enter independently; in particular, if one buyer is admitted to the auction then the other is not. Interestingly, introducing an entry cap raises the expected number of bidders from $2H(t^*(0, \phi^*)) = .6$ to $[1 - (1 - H(\bar{t}^*(\bar{\phi}^*)))^2] = .66$.

Scenario	(v, ϕ)	Equilibrium Threshold	Seller Revenue	Total Buyer Surplus	Social Surplus
(i)	$(0, 0)$	.3571	.06122 (100.00)	.04592 (100.00)	.10714 (100.00)
(ii)	$(.0972, 0)$	.3295	.07261 (118.60)	.02529 (55.06)	.09790 (91.37)
(iii)	$(0, .0750)$	.3250	.07500 (122.50)	.02250 (49.00)	.09750 (91.00)
(iv)	$(0, .1443)$ $(\bar{n} = 1)$	.3557	.09623 (157.16)	.03522 (76.70)	.13145 (122.68)

Table 1: Equilibrium Outcomes in Scenarios (i)-(iv).

6 Market Thickness

In this section we study the impact on seller revenue and social surplus of an increase in the number of buyers N. Consider a standard auction with a screening value and an admission fee both equal to zero, and assume that bidders' values are distributed uniformly on $[0, 1]$. The thick continuous curve in Figure 1 below shows seller revenue as a function of N when buyers have a homogenous entry cost of $c = 1/4$. Seller revenue decreases with N. (Levin and Smith (1994), Proposition 9, show that this is a general feature when entry costs are homogeneous.) The thin continuous curve in Figure 1 shows seller revenue when entry costs are distributed uniformly on $[1/4, 1/2]$. Seller revenue increases with N. The two curves approach each other as N becomes large and seem to converge to a common limit.

Figure 1: Seller Revenue and the Number of Buyers.

That seller revenue increases with N when entry costs are heterogeneous is not a general feature; e.g., seller revenue and social surplus decrease from $N = 1$ to $N = 2$ when entry costs are uniformly distributed on $[.49, .5]$.[13] The convergence of seller revenue to a common limit observed in Figure 1, however, holds in general.

Proposition 9 establishes that as N grows large, a screening value and an admission fee both equal to zero asymptotically generates the same seller revenue and social surplus when all buyers have the same entry cost $c > 0$ as when the lower bound of buyers' heterogeneous entry costs is $\underline{c} = c$. Hence, despite the different comparative static properties of equilibrium with homogeneous and heterogeneous entry costs, the equilibrium outcomes are asymptotically the same.

For each integer N, write W_N^* ($\hat{W}_N^*$) for the constrained maximum social surplus when buyers have heterogeneous (homogeneous) entry costs. Also denote by Π_N^0 ($\hat{\Pi}_N^0$) seller revenue in a standard auction with a screening value and admission fee both equal to zero when buyers have heterogeneous (homogeneous) entry costs.

Proposition 9. *A screening value and an admission fee both equal to zero asymptotically generate the same seller revenue and social surplus whether buyers have*

[13] Introducing an additional buyer has two effects: it worsens the entry coordination problem, as in Levin and Smith (1994), but also favors a better entry cost selection. Which effect dominates depends on the distribution of entry costs.

homogenous or heterogeneous entry costs, so long as $c = \underline{c}$; i.e.,

$$\lim_{N\to\infty} \Pi_N^0 = \lim_{N\to\infty} W_N^* = \lim_{N\to\infty} \hat{\Pi}_N^0 = \lim_{N\to\infty} \hat{W}_N^* > 0.$$

Hence a screening value and an admission fee equal to zero asymptotically maximize seller revenue when buyers have heterogeneous entry costs.

Proposition 9 has several implications: when entry cost are heterogeneous, seller revenue is asymptotically invariant to changes in the distribution of entry costs that preserve the lower bound of its support. Seller revenue and social surplus coincide asymptotically, and hence total buyer surplus is asymptotically zero. Finally, seller revenue is asymptotically the same whether the screening value and the admission fee are both set equal to zero or whether they are set to maximize seller revenue.

Proposition 10 establishes that an entry cap and admission fee allows the seller to asymptotically capture the *unconstrained maximum social surplus*, $s(0, \bar{n}) - \bar{n}\underline{c}$, where $\bar{n} = n^*(\underline{c})$ is the socially optimal number of bidders when all buyers have the lowest possible entry cost $\underline{c}$. This is illustrated in Figure 1 where the thin dashed line shows seller revenue when entry costs are heterogeneous and distributed uniformly on $[1/4, 1/2]$, and there is an entry cap of $\bar{n} = 1$ and an optimal admission fee. Seller revenue asymptotically approaches 1/4, the unconstrained maximum social surplus. The thick dashed shows seller revenue when buyers have a homogeneous entry cost of $c = 1/4$, and there is an entry cap of $\bar{n} = 1$ and an optimal admission fee. Consistent with Proposition 7, seller revenue is constant in N and equal to the unconstrained maximum social surplus.

Proposition 10. *An entry cap $\bar{n} = n^*(\underline{c})$, a revenue maximizing admission fee, and a zero screening value allows the seller to asymptotically capture the unconstrained maximum social surplus, $s(0, \bar{n}) - \bar{n}\underline{c}$.*

By Proposition 8, the introduction of the entry cap $\bar{n} = n^*(\underline{c})$ increases seller revenue. Proposition 10 implies that the revenue advantage of an entry cap persists asymptotically, i.e., $s(0, \bar{n}) - \bar{n}c > \lim_{N\to\infty} \Pi_N^0$. To see why this holds, first observe that for any fixed N we have $s(0, \bar{n}) - \bar{n}\underline{c} > \hat{W}_N^*$ and, since $\hat{W}_N^*$ decreases with N (by Levin and Smith (1994)), then $s(0, \bar{n}) - \bar{n}\underline{c} > \lim_{N\to\infty} \hat{W}_N^*$. Hence $s(0, \bar{n}) - \bar{n}\underline{c} > \lim_{N\to\infty} \Pi_N^0$ by Proposition 9.

An interesting case not covered by propositions 9 and 10 is when the lower bound of the support of entry costs is zero, i.e., $\underline{c} = 0$. Proposition 11 establishes that if values are uniformly distributed, then in a standard auction with a screening value and an admission fee both equal to zero, seller revenue and social surplus are asymptotically equal to $\bar{v}$ (the asymptotic maximum gross social surplus). An immediate implication of this result is that the total entry costs incurred by buyers, as well as total buyer surplus, are asymptotically zero. More significantly, seller revenue is the unconstrained maximum social surplus without screening buyers by entry costs or bidders by values, and without capping the number of entrants.

Proposition 11. *If $\underline{c} = 0$ and values are distributed uniformly on $[0, \bar{v}]$, then a screening value and an admission fee both equal to zero asymptotically generate a seller revenue and social surplus equal to $\bar{v}$, i.e.,*

$$\lim_{N\to\infty} \Pi_N^0 = \lim_{N\to\infty} W_N^* = \bar{v}.$$

Hence a screening value and an admission fee equal to zero asymptotically maximize seller revenue.

7 Conclusions

The results obtained when entry costs are homogeneous, namely that a standard auction realizes the maximum social surplus, and that this surplus is captured by the seller without screening bidders by value, is not robust to the introduction of heterogeneity in entry costs. In the generic case of heterogeneous entry costs, we show that maximizing seller revenue entails screening bidders by values or by entry costs if it is feasible, thereby inducing less entry than is socially optimal (and generating ex-post inefficiencies when screening bidders by value). In addition, whether entry costs are homogeneous or heterogeneous, an admission fee combined with an entry cap that appropriately trade off competition and surplus creation generates more revenue. As the number of buyers grows large, asymptotic seller revenue depends only on the lower bound of entry costs $\underline{c}$ and is the same as when entry costs are homogeneous and equal to $\underline{c}$, i.e., asymptotically there is no advantage to screen buyers by entry cost or values. However, the revenue advantage of an entry cap persists asymptotically so long as the lower bound of entry costs is positive.

8 Appendix

Proof of Proposition 1: For $n = 1$ we have

$$u(0,1) = \int_0^{\bar{v}} y f(y) dy = E(V_{(1)}) = s(0,1).$$

For $n > 1$, by interchanging the order of integration we obtain

$$\begin{aligned} u(0,n) &= \int_0^{\bar{v}} \left(\int_0^y F(x)^{n-1} dx \right) f(y) dy \\ &= \int_0^{\bar{v}} \left(\int_x^{\bar{v}} f(y) dy \right) F(x)^{n-1} dx \\ &= \int_0^{\bar{v}} (1 - F(x)) F(x)^{n-1} dx. \end{aligned}$$

Integrating by parts we get

$$\begin{aligned} \int_0^{\bar{v}} F(x)^n dx &= x F^n(x)|_0^{\bar{v}} - \int_0^{\bar{v}} n x F(x)^{n-1} f(x) dx \\ &= \bar{v} - E\left(V_{(n)}\right). \end{aligned}$$

Hence

$$\begin{aligned} u(0,n) &= \int_0^{\bar{v}} F(x)^{n-1} dx - \int_0^{\bar{v}} F(x)^n dx \\ &= \left(\bar{v} - E\left(V_{(n-1)}\right)\right) - \left(\bar{v} - E\left(V_{(n)}\right)\right) \\ &= s(0,n) - s(0,n-1). \ \square \end{aligned}$$

Proof of Proposition 2: Consider a standard auction with a screening value $v \in [0, \bar{v}]$ and an admission fee $\phi \in \mathbb{R}$. We show that there is the unique symmetric entry equilibrium, $t^*(v, \phi)$.

Assume that $u(v,1) \leq \underline{c} + \phi$. Since $p_0^{N-1}(0) = 1$ and $p_n^{N-1}(0) = 0$ for $n > 0$, then

$$U(v,0) = \sum_{n=0}^{N-1} p_n^{N-1}(0) u(v, n+1) = u(v,1).$$

Since U is decreasing in p we have

$$U(v, H(t)) \leq U(v,0) = u(v,1) \leq \underline{c} + \phi \leq z + \phi$$

for all $t, z \in [\underline{c}, \bar{c}]$. Therefore in equilibrium no buyer enters, i.e., $t^*(v, \phi) = \underline{c}$ is the unique symmetric entry equilibrium.

Assume that $u(v,N) \geq \bar{c}+\phi$. Since $p_n^{N-1}(1)=0$ for $n<N-1$ and $p_{N-1}^{N-1}(1)=1$, then

$$U(v,1)=\sum_{n=0}^{N-1} p_n^{N-1}(1)u(v,n+1)=u(v,N).$$

Since U is decreasing in p we have

$$U(v,H(t)) \geq U(v,1)=u(v,N) \geq \bar{c}+\phi \geq z+\phi$$

for all $t,z \in [\underline{c},\bar{c}]$. Therefore in equilibrium every buyer enters, i.e., $t^*(v,\phi)=\bar{c}$ is the unique symmetric entry equilibrium.

Assume that $u(v,1)>\underline{c}+\phi$ and $u(v,N)<\bar{c}+\phi$. Then

$$U(v,H(\underline{c}))=U(v,0)=u(v,1)>\underline{c}+\phi,$$

and

$$U(v,H(\bar{c}))=U(v,1)=u(v,N)<\bar{c}+\phi.$$

Since $U(v,H(\cdot))$ is continuous and decreasing on $[\underline{c},\bar{c}]$ (because $U(v,p)$ is decreasing and continuous in p and H is continuous and increasing in t) there is a unique $t^*(v,\phi) \in (\underline{c},\bar{c})$ solving the equation (3), $U(v,H(t))=t+\phi$. Hence $U(v,H(t^*(v,\phi)))>z+\phi$ implies $t^*(v,\phi)>z$, and $U(v,H(t^*(v,\phi)))<z+\phi$ implies $t^*(v,\phi)<z$, and therefore $t^*(v,\phi)$ is a symmetric entry equilibrium. To see that $t^*(v,\phi)$ is the unique symmetric entry equilibrium, note that for $\bar{t} \in [\underline{c},t^*(v,\phi))$ and $z \in (\bar{t},t^*(v,\phi))$ we have

$$U(v,H(\bar{t}))>U(v,H(t^*(v,\phi)))=t^*(v,\phi)+\phi>z+\phi.$$

Hence $\bar{t}$ is not a symmetric entry equilibrium. An analogous argument establishes that no $\bar{t} \in (t^*(v,\phi),\bar{c}]$ is a symmetric entry equilibrium either.

Since $U(v,p)$ is continuous in v (because each $u(\cdot,n)$ for $n \in \{1,...,N\}$ is continuous), then $t^*(v,\phi)$ is also continuous.

Finally, we show that $t^*(v,\phi)$ is decreasing in v and ϕ. Differentiating (3) implicitly and noticing that $U(v,p)$ is decreasing in both v and p yields

$$\frac{\partial t^*}{\partial \phi}=-\left(1-\frac{\partial U}{\partial p}h(t)\right)^{-1}<0,$$

and

$$\frac{\partial t^*}{\partial v}=\frac{\partial U}{\partial v}\left(1-\frac{\partial U}{\partial p}h(t)\right)^{-1}=-\frac{\partial U}{\partial v}\left(\frac{\partial t^*}{\partial \phi}\right)<0. \square$$

The following lemma is key in proving Proposition 3.

Lemma 1. *$W^* = W(0, t^W)$ where $t^W \in (\underline{c}, \bar{c})$ uniquely solves $U(0, H(t)) - t = 0$.*

Proof: Since $W(v,t)$ is decreasing in v, then $W^* = \max_{(v,t)\in[0,\omega]\times[\underline{c},\bar{c}]} W(v,t) = \max_{t\in[\underline{c},\bar{c}]} W(0,t)$. We have

$$\frac{dW(0,t)}{dt} = \sum_{n=1}^{N} \frac{dp_n^N(H(t))}{dt} s(0,n) - Nth(t).$$

Writing p_n^N for $p_n^N(H(t))$, we have

$$\frac{dp_n^N(H(t))}{dt} = N(p_{n-1}^{N-1} - p_n^{N-1})h(t),$$

for $n \leq N-1$, and

$$\frac{dp_N^N(H(t))}{dt} = Np_{N-1}^{N-1}h(t).$$

Substituting these expressions and using Proposition 1, we have

$$\begin{aligned} \frac{dW(0,t)}{dt} &= Nh(t)\left(p_{N-1}^{N-1}s(0,N) + \sum_{n=1}^{N-1}(p_{n-1}^{N-1} - p_n^{N-1})s(0,n) - t\right) \\ &= Nh(t)\left(\sum_{n=0}^{N-1} p_n^N u(0,n+1) - t\right) \\ &= Nh(t)\left(U(0,H(t)) - t\right). \end{aligned}$$

By assumption, we have $U(0, H(\underline{c})) - \underline{c} = U(0,0) - \underline{c} = u(0,1) - \underline{c} > 0$, and $U(0, H(\bar{c})) - \bar{c} = U(0,1) - \bar{c} = u(0,N) - \bar{c} < 0$. Since U is continuous and decreasing in p there is a unique $t^W \in (\underline{c}, \bar{c})$ such that $U(0,H(t)) - t = 0$. Moreover, since $h(t) > 0$ on $[\underline{c}, \bar{c}]$, then $dW(0,t)/dt > 0$ for $t \in [\underline{c}, t^W)$ and $dW(0,t)/dt < 0$ for $t \in (t^W, \bar{c}]$. Hence t^W is the unique maximizer of $W(0,t)$ on $[\underline{c}, \bar{c}]$. $\square$

Proof of Proposition 3: Proposition 3 follows directly from Lemma 1 by simply noting that the equation $U(0, H(t)) - t = 0$ is identical to equation (3) for $v = \phi = 0$; i.e., $t^W = t^*(0,0)$. Hence $W^* = W(0, t^*(0,0))$. $\square$

Lemmas 2, 3 and 4 are useful in the proof of Proposition 5.

Lemma 2. $\left.\frac{d\Pi(v,H(t^*(v,0)))}{dv}\right|_{v=0} > 0.$

Proof: For $\phi = 0$, differentiating equation (7) with respect to v we have

$$\left.\frac{d\Pi(v, H(t^*(v,0)))}{dv}\right|_{v=0} = \left.\frac{dW(v, t^*(v,0))}{dv}\right|_{v=0} - NH(t^*(v,0))\left.\frac{dt^*(v,0)}{dv}\right|_{v=0}.$$

Since $W(v, t^*(v,0))$ is maximized at $v = 0$ by Proposition 3, we have

$$\frac{\partial W(0, t^*(0,0))}{\partial t} = 0.$$

Taking the right derivative of $W(v,t)$ with respect to v at $v = 0$ we get

$$\left.\frac{\partial W(v,t)}{\partial v}\right|_{v=0} = 0.$$

Then we have

$$\left.\frac{dW(v, t^*(v,0))}{dv}\right|_{v=0} = \frac{\partial W(0, t^*(0,0))}{\partial v} + \frac{\partial W(0, t^*(0,0))}{\partial t}\frac{dt^*}{dv} = 0.$$

Since $t^*(0,0) = t^W$ by Proposition 3 and $t^W \in (\underline{c}, \bar{c})$ by Lemma 1, then $t^*(v,0)$ is decreasing at $v = 0$, and therefore

$$\left.\frac{d\Pi(v, H(t^*(v,0)))}{dv}\right|_{v=0} = -NH(t^*(0,0))\frac{dt^*(0,0)}{dv} > 0. \ \square$$

Recall that v^F, the solution to the equation $v = (1-F(v))/f(v)$, maximizes $\pi(\cdot, n)$ on $[0, \bar{v}]$ – see Proposition 5 in Riley and Samuelson (1981).

Lemma 3. *If $t^*(v^F, 0) > \underline{c}$, then $\Pi(v^F, H(t^*(v^F,0))) > \Pi(v, H(t^*(v,0)))$ for $v > v^F$.*

Proof: For $v > v^F$, then $t^*(v^F,0) > \underline{c}$ implies $t^*(v^F,0) \geq t^*(v,0)$ by Proposition 2. Hence the *c.d.f.* of the binomial $B(N, H(t^*(v^F,0)))$ first order stochastically dominates the *c.d.f.* of the binomial $B(N, H(t^*(v,0)))$. Since π is strictly increasing in n and $\pi(v^F, n) > \pi(v,n)$ for $v \in (v^F, \bar{v}]$, we have

$$\begin{aligned}
\Pi(v^F, H(t^*(v^F,0))) &= \sum_{n=1}^{N} p_n^N(H(t^*(v^F,0)))\pi(v^F,n) \\
&> \sum_{n=1}^{N} p_n^N(H(t^*(v,0)))\pi(v^F,n) \\
&> \sum_{n=1}^{N} p_n^N(H(t^*(v,0)))\pi(v,n) \\
&= \Pi(v, H(t^*(v,0))). \ \square
\end{aligned}$$

Lemma 4. *If* $t^*(v^F,0) \in (\underline{c}, \bar{c})$, *then* $\left.\frac{d\Pi(v,H(t^*(v,0)))}{dv}\right|_{v=v^F} < 0$.

Proof: Assume that $t^*(v^F,0) \in (\underline{c}, \bar{c})$. We have

$$\begin{aligned} \left.\frac{d\Pi(v,H(t^*(v,0)))}{dv}\right|_{v=v^F} &= \sum_{n=1}^{N} \left.\frac{dp_n^N(H(t^*(v,0)))}{dv}\right|_{v=v^F} \pi(v^F,n) \\ &\quad + \sum_{n=1}^{N} p_n^N(H(t^*(v^F,0))) \left.\frac{d\pi(v,n)}{dv}\right|_{v=v^F}. \end{aligned}$$

For all $n \geq 1$, since v^F maximizes $\pi(\cdot,n) \in [0,\bar{v}]$, we have

$$\left.\frac{d\pi(v,n)}{dv}\right|_{v=v^F} = 0.$$

Hence

$$\begin{aligned} \left.\frac{d\Pi(v,H(t^*(v,0)))}{dv}\right|_{v=v^F} &= \sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=H(t^*(v^F,0))} \left.\frac{dH(t)}{dt}\right|_{t=t^*(v^F,0)} \left.\frac{dt^*(v,0)}{dv}\right|_{v=v^F} \pi(v^F,n) \\ &= h(t^*(v^F,0))\frac{dt^*(v^F,0)}{dv} \sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=H(t^*(v^F,0))} \pi(v^F,n). \end{aligned}$$

In this expression, $h(t^*(v^F,0)) > 0$ and $\frac{dt^*(v^F,0)}{dv} < 0$ (by Proposition 2). The term

$$\sum_{n=1}^{N} \left.\frac{dp_n^N(p)}{dp}\right|_{p=H(t^*(v^F,0))} \pi(v^F,n),$$

is positive: an increase in the binomial probability induces a new binomial distribution whose *c.d.f.* first order stochastically dominates the *c.d.f.* of $B(N,H(t^*(v^F,0)))$ which, because π is increasing in n, increases seller revenue. Therefore

$$\left.\frac{d\Pi(v,H(t^*(v,0)))}{dv}\right|_{v=v^F} < 0. \ \square$$

Proof of Proposition 5: Since $\phi = 0$, then for $v \in [0,\bar{v}]$ seller revenue is $\Pi(v,H(t^*(v,0)))$, which is continuous on $[0,\bar{v}]$. Hence an optimal screening value v^* exists. We have $0 < v^*$ by Lemma 2. We show that $v < v^F$. Assume that $t^*(v^F,0) = \underline{c}$; then for all $v \in [v^F,\bar{v}]$ we have

$$\Pi(v,H(t^*(v,0))) = 0 < \Pi(0,H(t^*(0,0))) < \Pi(v^*,H(t^*(v^*,0))).$$

Hence $v^* < v^F$. Assume that $t^*(v^F, 0) > \underline{c}$. Then $v^* \leq v^F$ by Lemma 3. Since $t^*(0,0) = t^W$ by Proposition 3 and $t^W \in (\underline{c}, \bar{c})$ by Lemma 1, then $t^*(v, 0)$ is decreasing at $v = 0$ by Proposition 2. Hence $v^F > 0$ implies $t^*(v^F, 0) < t^*(0,0) < \bar{c}$. Hence $t^*(v^F, 0) \in (\underline{c}, \bar{c})$, and Lemma 4 implies $v^* \neq v^F$. Hence $v^* < v^F$. Since $v^* \in (0, \bar{v})$, then it solves equation (8). $\square$

Proof of Proposition 6: Assume that (v^*, ϕ^*) maximize seller revenue. We show that $v^* = 0$ and $\phi^* > 0$.

We begin by showing that $t^*(v^*, \phi^*) > \underline{c}$; i.e., there is entry. Since seller revenue is positive for $(v, \phi) = (0, 0)$, and seller revenue is zero when there is no entry, i.e., when the equilibrium threshold is $\underline{c}$, then $t^*(v^*, \phi^*) > \underline{c}$.

We prove now that $v^* = 0$. Assume that $v^* > 0$, and define

$$\hat{\phi} = U(0, H(\hat{t})) - \hat{t},$$

where $\hat{t} = t(v^*, \phi^*) > \underline{c}$. Then

$$U(0, H(\hat{t})) = \hat{t} + \hat{\phi}.$$

Hence $t^*(0, \hat{\phi}) = \hat{t} = t^*(v^*, \phi^*)$, i.e., the equilibrium threshold is the same for $(0, \hat{\phi})$ and for (v^*, ϕ^*), and therefore total buyer surplus is also the same. Social surplus is greater for $(0, \hat{\phi})$ than for (v^*, ϕ^*) , since for $v = 0$ the auction is ex-post efficient, whereas for $v^* > 0$ it is not. Thus, seller revenue is greater for $(0, \hat{\phi})$, contradicting that (v^*, ϕ^*) maximizes seller revenue.

We show that $\phi^* \neq 0$. Since $v^* = 0$, if $\phi^* = 0$, then the maximum seller revenue is $\Pi(0, H(t^*(0, 0)))$. By Proposition 5, however, when no admission fee is feasible (i.e., when $\phi = 0$) the revenue maximizing screening value is positive; i.e., seller revenue with a positive screening value is larger than $\Pi(0, H(t^*(0, 0)))$. Hence $\phi^* \neq 0$.

We show that $\phi^* \geq 0$. Assume that $\phi < 0$. Since social surplus is uniquely maximized at $(v, \phi) = (0, 0)$ by Proposition 3, raising the admission fee to zero, while maintaining the screening value equal to zero, increases social surplus, and does not increase buyer surplus (because the entry threshold is weakly decreasing in ϕ). Hence seller revenue increases; i.e., $\phi < 0$ does not maximize seller revenue.

Finally, the existence of an optimal admission fee ϕ^* is guaranteed since for $v = 0$ seller revenue, given in equation (7), is continuous on $[0, \bar{\phi}]$, where $\bar{\phi} = u(0, 1) - \underline{c}$, and it is zero for $\phi > \bar{\phi}$ as shown in the proof of Proposition 2 above. Moreover, $\phi^* \in (0, \bar{\phi})$ and hence must satisfy equation (9). $\square$

Proof of Proposition 8: Consider a standard auction with a screening value equal to zero, an admission fee ϕ, and an entry cap $\bar{n} \geq 1$. We first show that the entry game has a unique symmetric equilibrium threshold. For $z, t \in [\underline{c}, \bar{c}]$ define

$$\varphi(\phi, z, t) := \alpha(H(t)) \left(\bar{U}(H(t)) - (z + \phi)\right);$$

i.e.,

$$\begin{aligned}\varphi(\phi, z, t) &= \sum_{n=0}^{\bar{n}-1} p_n^{N-1}(H(t)) \left(u(0, n+1) - z - \phi\right) \\ &\quad + \sum_{n=\bar{n}}^{N-1} p_n^{N-1}(H(t)) \frac{\bar{n}}{n+1} \left(u(0, \bar{n}) - z - \phi\right) \\ &= \sum_{n=0}^{N-1} p_n^{N-1}(H(t)) \bar{u}(\phi, z, n+1),\end{aligned}$$

where

$$\bar{u}(\phi, z, n) = \begin{cases} u(0, n) - z - \phi & \text{if } n \leq \bar{n} \\ \frac{\bar{n}}{n+1} \left(u(0, \bar{n}) - z - \phi\right) & \text{if } n > \bar{n}. \end{cases}$$

For each $(\phi, z) \in \mathbb{R} \times [\underline{c}, \bar{c}]$, we have that $\bar{u}$ is decreasing in n. Thus, $\varphi(\phi, z, t)$ is decreasing in t because for $t' > t$, $B(N-1, H(t'))$ first order stochastically dominates $B(N-1, H(t))$. Also for $z, z' \in [\underline{c}, \bar{c}]$ we have

$$\varphi(\phi, z', t) - \varphi(\phi, z, t) = -\alpha(H(t))(z' - z).$$

Define $\psi(\phi, t) := \varphi(\phi, t, t)$. We show that ψ is decreasing in t. Let $t' > t$. Then

$$\begin{aligned}\psi(\phi, t') - \psi(\phi, t) &= \varphi(\phi, t', t') - \varphi(\phi, t, t) \\ &= \varphi(\phi, t, t') - \varphi(\phi, t, t) + \varphi(\phi, t', t') - \varphi(\phi, t, t') \\ &= \varphi(\phi, t, t') - \varphi(\phi, t, t) - \alpha(H(t'))(t' - t) < 0.\end{aligned}$$

Since α is decreasing in p and $\alpha(H(\bar{c})) = \alpha(1) = \bar{n}/N > 0$, we have $\alpha(H(t)) > 0$

for all $t \in [\underline{c}, \bar{c}]$. Let $t \in [\underline{c}, \bar{c}]$. Then for $t', z \in [t, \bar{c}]$ we have

$$\begin{aligned} \psi(\phi, t) &= \varphi(\phi, t, t) \\ &\geq \varphi(\phi, t, t') \\ &= \alpha(H(t')) \left(\bar{U}(H(t')) - (t + \phi)\right) \\ &\geq \alpha(H(t')) \left(\bar{U}(H(t')) - (z + \phi)\right). \end{aligned}$$

If $\psi(\phi, \underline{c}) < 0$, then $\bar{U}(H(t')) - (z + \phi) < 0$ for all $t', z \in [\underline{c}, \bar{c}]$, and therefore $\bar{t}^*(\phi) = \underline{c}$ is the unique equilibrium. Likewise, if $\psi(\phi, \bar{c}) > 0$, then $\bar{t}^*(\phi) = \bar{c}$ is the unique equilibrium.

Finally, if $\psi(\phi, \underline{c}) > 0 > \psi(\phi, \bar{c})$, since $\psi(\phi, t)$ is decreasing in t, then there is a unique $\bar{t} \in (\underline{c}, \bar{c})$ such that $\psi(\phi, \bar{t}) = 0$; hence $\alpha(H(\bar{t})) > 0$ implies $\bar{U}(H(\bar{t})) = \bar{t} + \phi$. Moreover, $\bar{U}(H(\bar{t})) < z + \phi$ for all $z \in (\bar{t}, \bar{c}]$ and $\bar{U}(H(\bar{t})) > z + \phi$ for all $z \in [\underline{c}, \bar{t})$. Therefore $\bar{t}$ is an equilibrium. Let $t \in [\underline{c}, \bar{t})$. We have $\psi(\phi, t) > 0$, i.e., $\bar{U}(H(t)) > t + \phi$. Hence for $z = t + \frac{1}{2}\left(\bar{U}(H(t)) - t - \phi\right)$, we have $\bar{U}(H(t)) > z + \phi$ and $z > t$, and therefore t is not an equilibrium. Likewise, no $t \in (\bar{t}, \bar{c}]$ is an equilibrium either. Hence $\bar{t}^*(\phi) = \bar{t}$ is the unique equilibrium.

We establish Proposition 8 by showing that a standard auction with an entry cap $\bar{n} = n^*(\underline{c})$, a screening value of zero, and the admission fee

$$\bar{\phi} = \phi^* + \bar{U}(H(t^*(0, \phi^*))) - U(0, H(t^*(0, \phi^*))),$$

generates more seller revenue than the auction with no entry cap, and a revenue maximizing admission fee ϕ^* and screening value $v = 0$. This is established by showing that total buyer surplus in the auction with entry cap $\bar{n} = n^*(\underline{c})$ and admission fee $\bar{\phi}$, denoted by $\bar{B}$, is less than total buyer surplus in the auction with no entry cap and admission fee ϕ^*, denoted by B, whereas social surplus in the former, denoted by $\bar{W}$, is greater than in the latter; i.e., $\bar{W} > W(0, t^*(0, \phi^*))$. We have

$$\begin{aligned} \psi(\bar{\phi}, t^*(0, \phi^*)) &= \alpha(H(t^*(0, \phi^*))) \left(\bar{U}(H(t^*(0, \phi^*))) - \bar{\phi} - t^*(0, \phi^*)\right) \\ &= \alpha(H(t^*(0, \phi^*))) \left(U(0, H(t^*(0, \phi^*))) - \phi^* - t^*(0, \phi^*)\right) \\ &= 0. \end{aligned}$$

Hence $t^*(0, \phi^*) = \bar{t}^*(\bar{\phi})$; i.e., the equilibrium threshold is the same in the auction with no entry cap and admission fee ϕ^* as in the auction with entry cap $\bar{n}$ and admission fee $\bar{\phi}$. Write $\bar{t} = t^*(0, \phi^*) = \bar{t}^*(\bar{\phi})$.

We show that $\bar{B} < B$. In the auction with entry cap $\bar{n} = n^*(\underline{c})$ and admission fee $\bar{\phi}$ the (ex-ante) surplus of a bidder whose entry cost is $z < \bar{t}$ is equal to the probability of being admitted to the auction, $\alpha(H(\bar{t}))$, times her payoff conditional on being admitted, $\bar{U}(H(\bar{t})) - \bar{\phi} - z$. We have

$$\begin{aligned}
\bar{B} &= N \int_{\underline{c}}^{\bar{t}} \alpha(H(\bar{t})) \left(\bar{U}(H(\bar{t})) - \bar{\phi} - z\right) dH(z) \\
&< N \int_{\underline{c}}^{\bar{t}} \left(\bar{U}(H(\bar{t})) - \bar{\phi} - z\right) dH(z) \\
&= N \int_{\underline{c}}^{\bar{t}} (U(0, H(\bar{t})) - \phi^* - z) dH(z) \\
&= B,
\end{aligned}$$

where we use the equation $\bar{U}(H(\bar{t})) - \bar{\phi} = U(0, H(\bar{t})) - \phi^*$, and where the inequality holds since $N > \bar{n}$ and $\alpha(H(\bar{t})) < \alpha(H(\underline{c})) = \alpha(0) = 1$.

Finally, we show that $\bar{W} > W(0, \bar{t})$. By Proposition 1 we have

$$s(0, n) = \sum_{k=1}^{n} u(0, k).$$

Since $u(0, n) - E[z|z \leq \bar{t}] \leq u(0, \bar{n} + 1) - \underline{c} < 0$ for $n \geq \bar{n} + 1$, we have

$$\begin{aligned}
\bar{W} &= \sum_{n=1}^{\bar{n}} p_n^N(H(\bar{t}))\,(s(0, n) - nE[z|z \leq \bar{t}]) + \sum_{n=\bar{n}+1}^{N} p_n^N(H(\bar{t}))\,(s(0, \bar{n}) - \bar{n}E[z|z \leq \bar{t}]) \\
&= \sum_{n=1}^{\bar{n}} p_n^N(H(\bar{t}))\,(s(0, n) - nE[z|z \leq \bar{t}]) + \sum_{n=\bar{n}+1}^{N} p_n^N(H(\bar{t})) \sum_{k=1}^{\bar{n}} (u(0, k) - E[z|z \leq \bar{t}]) \\
&> \sum_{n=1}^{\bar{n}} p_n^N(H(\bar{t}))\,(s(0, n) - nE[z|z \leq \bar{t}])) + \sum_{n=\bar{n}+1}^{N} p_n^N(H(\bar{t})) \sum_{k=1}^{n} (u(0, k) - E[z|z \leq \bar{t}]) \\
&= \sum_{n=1}^{N} p_n^N(H(\bar{t}))(s(0, n) - n\frac{c(\bar{t})}{H(\bar{t})}) \\
&= \sum_{n=1}^{N} p_n^N(H(\bar{t}))s(0, n) - Nc(\bar{t}) \\
&= W(0, \bar{t}),
\end{aligned}$$

since the sum immediately after the inequality includes the negative terms $u(0, k) - E[z|z \leq \bar{t}]$ for $k > \bar{n}$ and since $p_n^N(H(\bar{t})) > 0$ for $n \in \{\bar{n} + 1, \ldots, N\}$. □

Proof of Proposition 9: Assume $c = \underline{c} > 0$. By Proposition 9 in Levin and Smith (1994) the sequence $\{\hat{W}_N^*\} \subset [0, \bar{v}]$ is decreasing. Hence it has a limit. Moreover, since $\hat{\Pi}_N^0 = \hat{W}_N^*$ for each N, we have

$$\lim_{N\to\infty} \hat{\Pi}_N^0 = \lim_{N\to\infty} \hat{W}_N^*.$$

For each N we use the notation Π_N, U_N, S_N, t_N^*, W_N, and to refer to the functions Π, U, S, t^*, W defined in sections 2 and 4 for fixed N. Also we write p_N^* for the equilibrium entry probability when entry costs are homogeneous, and the screening value and admission fee are both equal to zero.

By Lemma 1 $\hat{t} \in (\underline{c}, \bar{c})$, and by Proposition 3 $t_N^*(0,0) = \hat{t}$. Hence $E[z|z \leq t_N^*(0,0)] > \underline{c} = c$. Again by Proposition 3, $W_N^* = W_N(0, t_N^*(0,0))$. We have

$$\begin{aligned}
\hat{W}_N^* &= \max_{p\in[0,1]} S_N(0,p) - Npc \\
&\geq S_N(0, H(t_N^*(0,0))) - NH(t_N^*(0,0))c \\
&> S_N(0, H(t_N^*(0,0))) - NH(t_N^*(0,0))E(z \mid z \leq t_N^*(0,0)) \\
&= W_N^*;
\end{aligned}$$

i.e., for each N, the constrained maximum social surplus is greater when entry costs are homogeneous than when they are heterogeneous.

We show

$$\lim_{N\to\infty} W_N^* = \lim_{N\to\infty} \hat{W}_N^*.$$

For each N, let $\hat{t}_N \in [\underline{c}, \bar{c}]$ be such that $H(\hat{t}_N) = p_N^*$. Then

$$W_N(0, \hat{t}_N) = S_N(0, p_N^*) - Np_N^* E(z \mid z \leq \hat{t}_N).$$

Since $\hat{W}_N^* \geq 0$ and $S(0, p_N^*) \leq \bar{v}$, then $0 \leq Np_N^* \leq \bar{v}/c$ for each N, and hence $\lim_{N\to\infty} p_N^* = \lim_{N\to\infty} H(\hat{t}_N) = 0$. Therefore $\lim_{N\to\infty} \hat{t}_N = \underline{c} = \lim_{N\to\infty} E(z \mid z \leq \hat{t}_N)$. Since

$$0 < \hat{W}_N^* - W_N(0, \hat{t}_N) = Np_N^*(E(z \mid z \leq \hat{t}_N) - \underline{c}),$$

and $\{Np_N^*\}$ is a bounded sequence, then

$$\lim_{N\to\infty} (\hat{W}_N^* - W_N(0, \hat{t}_N)) = 0,$$

and therefore

$$\lim_{N\to\infty} W_N(0,\hat{t}_N) = \lim_{N\to\infty} \hat{W}_N^* - \lim_{N\to\infty}(\hat{W}_N^* - W_N(0,\hat{t}_N)) = \lim_{N\to\infty} \hat{W}_N^*.$$

By Proposition 3 and the inequality above, we have

$$W_N(0,\hat{t}_N) \le W_N^* < \hat{W}_N^*$$

for all N. Hence

$$\lim_{N\to\infty} W_N^* = \lim_{N\to\infty} \hat{W}_N^*.$$

Next we show that $\lim_{N\to\infty}\Pi_N^0 = \lim_{N\to\infty} W_N^*$. Since $c = \underline{c}$, we have

$$U_N(0, H(t_N^*(0,0))) = t_N^*(0,0) \ge \underline{c} = U_N(0, p_N^*).$$

Hence $0 \le H(t_N^*(0,0)) \le p_N^*$ for all N. Since $\lim_{N\to\infty} p_N^* = 0$ then $\lim_{N\to\infty} H(t_N^*(0,0)) = 0$ and

$$\lim_{N\to\infty} t_N^*(0,0) = \lim_{N\to\infty} E(z \mid z \le t_N^*(0,0)) = \underline{c}.$$

Further, since $0 \le Np_N^* \le \bar{v}/c$ (as shown above), then $0 \le NH(t_N^*(0,0)) \le Np_N^* \le \bar{v}/c$; i.e., the sequence $\{NH(t_N^*(0,0))\}$ is bounded. Hence the asymptotic total buyer surplus is

$$\lim_{N\to\infty} NH(t_N^*(0,0))[t_N^*(0,0) - E(z \mid z \le t_N^*(0,0))] = 0.$$

Thus the asymptotic seller revenue is $\lim_{N\to\infty}\Pi_N^0 = \lim_{N\to\infty} W_N^*$. □

Proof of Proposition 10: For each N we denote by $\bar{U}_N$, α_N, and $\bar{t}_N^*$ the functions $\bar{U}$, α, and $\bar{t}^*$ defined in Section 4 for an auction with an entry cap $\bar{n} = n^*(\underline{c})$ and fixed N. Let $\varepsilon > 0$ arbitrary, and let the admission fee be $\bar{\phi} = u(0,\bar{n}) - \underline{c} - \frac{\varepsilon}{2\bar{n}}$. We show that for N sufficiently large seller revenue is greater than $s(0,\bar{n}) - \bar{n}\underline{c} - \varepsilon$, which establishes Proposition 10. We have

$$\begin{aligned}
\bar{U}_N(H(\bar{t}_N^*)) &= \sum_{n=0}^{\bar{n}-1} \frac{p_n^{N-1}(H(\bar{t}_N^*))}{\alpha_N(H(\bar{t}_N^*))} u(0,n+1) + \sum_{n=\bar{n}}^{N-1} \frac{p_n^{N-1}(H(\bar{t}_N^*))}{\alpha_N(H(\bar{t}_N^*))}\frac{\bar{n}}{n+1} u(0,\bar{n}) \\
&\ge \sum_{n=0}^{\bar{n}-1} \frac{p_n^{N-1}(H(\bar{t}_N^*))}{\alpha_N(H(\bar{t}_N^*))} u(0,\bar{n}) + \sum_{n=\bar{n}}^{N-1} \frac{p_n^{N-1}(H(\bar{t}_N^*))}{\alpha_N(H(\bar{t}_N^*))}\frac{\bar{n}}{n+1} u(0,\bar{n}) \\
&= u(0,\bar{n}),
\end{aligned}$$

where the inequality follows since $u(0,n)$ is decreasing in n. Hence for $z \in [\underline{c}, \underline{c} + \frac{\varepsilon}{2\bar{n}})$ we have

$$\bar{U}_N(H(\bar{t}_N^*)) - z - \bar{\phi} \geq u(0,\bar{n}) - z - \bar{\phi} > 0,$$

i.e., in equilibrium a buyer whose entry cost is $z \in [\underline{c}, \underline{c} + \frac{\varepsilon}{2\bar{n}})$ enters. Therefore $\bar{t}_N^* \geq \underline{c} + \frac{\varepsilon}{2\bar{n}}$ and $H(\bar{t}_N^*) \geq H(\underline{c} + \frac{\varepsilon}{2\bar{n}}) > 0$.

The equilibrium probability of at least $\bar{n}$ applicants is $\sum_{n=\bar{n}}^{N} p_n^N(H(\bar{t}_N^*))$. Since

$$\sum_{n=\bar{n}}^{N} p_n^N(H(\underline{c} + \frac{\varepsilon}{2\bar{n}})) \leq \sum_{n=\bar{n}}^{N} p_n^N(H(\bar{t}_N^*))$$

for each N, and $\lim_{N\to\infty} \sum_{n=\bar{n}}^{N} p_n^N(H(\underline{c} + \frac{\varepsilon}{2\bar{n}})) = 1$, then

$$\lim_{N\to\infty} \sum_{n=\bar{n}}^{N} p_n^N(H(\bar{t}_N^*)) = 1,$$

and

$$\lim_{N\to\infty} \sum_{n=0}^{\bar{n}-1} p_n^{N-1}(H(\bar{t}_N^*)) = 0.$$

Social surplus is

$$\bar{W}_N^* = \sum_{n=0}^{\bar{n}-1} p_n^N(H(\bar{t}_N^*))[s(0,n) - n\frac{c(\bar{t}_N^*)}{H(\bar{t}_N^*)}] + [s(0,\bar{n}) - \bar{n}\frac{c(\bar{t}_N^*)}{H(\bar{t}_N^*)}] \sum_{n=\bar{n}}^{N} p_n^N(H(\bar{t}_N^*)).$$

For each N, we can calculate total buyer surplus, $\bar{B}_N$, as

$$\begin{aligned}\bar{B}_N &= \sum_{n=1}^{\bar{n}-1} p_n^N(H(\bar{t}_N^*))n[u(0,n) - \frac{c(\bar{t}_N^*)}{H(\bar{t}_N^*)} - \phi] \\ &\quad + \bar{n}[u(0,\bar{n}) - \frac{c(\bar{t}_N^*)}{H(\bar{t}_N^*)} - \phi] \sum_{n=\bar{n}}^{N} p_n^N(H(\bar{t}_N^*)).\end{aligned}$$

Hence seller revenue is

$$\begin{aligned}\bar{W}_N^* - \bar{B}_N &= \sum_{n=0}^{\bar{n}-1} p_n^N(H(\bar{t}_N^*)) \left[s(0,n) - n\left(u(0,n) - \bar{\phi}\right)\right] \\ &\quad + \left[s(0,\bar{n}) - \bar{n}\left(u(0,\bar{n}) - \bar{\phi}\right)\right] \sum_{n=\bar{n}}^{N} p_n^N(H(\bar{t}_N^*)) \\ &= s(0,\bar{n}) - \bar{n}\underline{c} - \frac{\varepsilon}{2} - A_N,\end{aligned}$$

where

$$A_N = \sum_{n=0}^{\bar{n}-1} p_n^N(H(\bar{t}_N^*)) \left\{ \left[s(0,\bar{n}) - \bar{n} \left(u(0,\bar{n}) - \bar{\phi} \right) \right] - \left[s(0,n) - n \left(u(0,n) - \bar{\phi} \right) \right] \right\}.$$

Let $\bar{N}$ be sufficiently large that $A_N < \varepsilon/2$ for $N > \bar{N}$. Then for $N > \bar{N}$ we have

$$\bar{W}_N^* - \bar{B}_N \geq s(0,\bar{n}) - \bar{n}\underline{c} - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = s(0,\bar{n}) - \bar{n}\underline{c} - \varepsilon. \ \square$$

Proof of Proposition 11: Assume without loss of generality that $\bar{v} = 1$. We first establish that $\lim_{N\to\infty} W_N^* = 1$ by showing that for every $\varepsilon > 0$ there is $\bar{N}$ sufficiently large that $W_N^* > 1 - \varepsilon$ for all $N \geq \bar{N}$.

Fix $\varepsilon > 0$. Let λ be such that $1 - \frac{1}{\lambda}(1 - e^{-\lambda}) > 1 - \varepsilon$, i.e., $\frac{1}{\lambda}(1 - e^{-\lambda}) < \varepsilon$. Such a λ exists since $\lim_{\lambda\to\infty} \frac{1}{\lambda}(1 - e^{-\lambda}) = 0$. For each $N > \lambda$, let $t_N \in [0, \bar{c}]$ be such that $H(t_N) = \frac{\lambda}{N}$. Note t_N exists and is unique since H is continuous and increasing. Moreover, since $H(0) = 0$ and H is continuous, then $\lim_{N\to\infty} t_N = 0$.

Since values are uniformly distributed, then $s(0,n) = n/(n+1)$. We have

$$W_N(0, t_N) = \sum_{n=0}^{N} p_n^N(H(t_N)) \frac{n}{n+1} - N \int_0^{t_N} z dH(z).$$

Since $NH(t_N) = \lambda$ for all N and $\lim_{N\to\infty} t_N = 0$, we have

$$\begin{aligned} \lim_{N\to\infty} N \int_0^{t_N} z dH(z) &= \lim_{N\to\infty} NH(t_N) \int_0^{t_N} \frac{z}{H(t_N)} dH(z) \\ &= \lambda \lim_{N\to\infty} \int_0^{t_N} \frac{z}{H(t_N)} dH(z) \\ &= 0. \end{aligned}$$

Since the limit of a binomial distribution as N goes to infinity, holding $NH(t_N) = \lambda$ fixed, is the Poisson distribution, we have

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=0}^{N} p_n^N(H(t_N)) \frac{n}{n+1} &= \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \frac{n}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} (1 - \frac{1}{n+1}) \\ &= 1 - \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^{n+1}}{n!} \frac{1}{n+1}. \end{aligned}$$

Letting $k = n + 1$, i.e., $n = k - 1$ we have

$$1 - \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{e^{-\lambda}\lambda^{n+1}}{n!}\frac{1}{n+1} = 1 - \frac{1}{\lambda}(-e^{-\lambda} + \sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^{k}}{k!}) = 1 - \frac{1}{\lambda}(-e^{-\lambda} + 1).$$

Hence

$$\lim_{N\to\infty} W_N(0, t_N) = 1 - \frac{1}{\lambda}(-e^{-\lambda} + 1).$$

Let δ such that $0 < \delta < \varepsilon - \frac{1}{\lambda}(1 - e^{-\lambda})$ and $\bar{N}$ be sufficiently large that for all $N > \bar{N}$

$$W_N(0, t_N) \geq 1 - \frac{1}{\lambda}(1 - e^{-\lambda}) - \delta > 1 - \varepsilon.$$

By the definition of W_N^* we have

$$W_N^* \geq W_N(0, t_N) > 1 - \varepsilon$$

for all $N > \bar{N}$. Hence $\lim_{N\to\infty} W_N^* = 1$.

It remains to be shown that total buyer surplus is asymptotically zero. By Proposition 3 we have

$$W_N^* = W_N(0, (t_N^*(0,0))) = \sum_{n=0}^{N} p_n^N(H(t_N^*(0,0)))\frac{n}{n+1} - N\int_0^{t_N^*(0,0)} zdH(z).$$

Since $0 < \sum_{n=0}^{N} p_n^N(H(t_N^*(0,0)))\frac{n}{n+1} \leq 1$ and $N\int_0^{t_N^*(0,0)} zdH(z) > 0$ for all N, then $\lim_{N\to\infty} W_N^* = 1$ implies

$$\lim_{N\to\infty}\sum_{n=0}^{N} p_n^N(H(t_N^*(0,0)))\frac{n}{n+1} = 1.$$

Total buyer surplus satisfies

$$0 \leq N\int_0^{t_N^*(0,0)}(U_N(0, t^*(0,0)) - z)dH(z) < NH(t_N^*(0,0))U_N(0, t^*(0,0)).$$

Since values are distributed uniformly on $[0,1]$, then $u(0,n) = \frac{1}{n(n+1)}$ and

$$\begin{aligned} NH(t_N^*(0,0))U_N(0, t_N^*(0,0))) &= NH(t_N^*(0,0))\sum_{n=0}^{N-1} p_n^{N-1}(H(t_N^*(0,0)))u(0, n+1) \\ &= \sum_{n=1}^{N} p_n^N(H(t_N^*(0,0)))\frac{1}{n+1} \\ &< \sum_{n=0}^{N} p_n^N(H(t_N^*(0,0)))\frac{1}{n+1}. \end{aligned}$$

Using that $\frac{n}{n+1} = 1 - \frac{1}{n+1}$, we can write

$$\begin{aligned}\lim_{N\to\infty} \sum_{n=0}^{N} p_n^N (H(t_N^*(0,0))) \frac{n}{n+1} &= \lim_{N\to\infty} \sum_{n=0}^{N} p_n^N (H(t_N^*(0,0))) \left(1 - \frac{1}{n+1}\right) \\ &= 1 - \lim_{N\to\infty} \sum_{n=0}^{N} p_n^N (H(t_N^*(0,0))) \frac{1}{n+1},\end{aligned}$$

provided that this last limit exists. Since $\sum_{n=0}^{N} p_n^N (H(t_N^*(0,0))) \frac{1}{n+1} \in [0,1]$ for each N, and every convergent subsequence has a limit of zero, then the sequence itself has a limit of zero, i.e.,

$$\lim_{N\to\infty} \sum_{n=0}^{N} p_n^N (H(t_N^*(0,0))) \frac{1}{n+1} = 0.$$

Hence

$$\lim_{N\to\infty} NH(t_N^*(0,0)) t_N^*(0,0) = 0.$$

Therefore total buyer surplus is asymptotically zero, and by (7) seller revenue is asymptotically the entire social surplus, i.e., $\lim_{N\to\infty} \Pi_N(0, t_N^*(0,0))) = 1$. □

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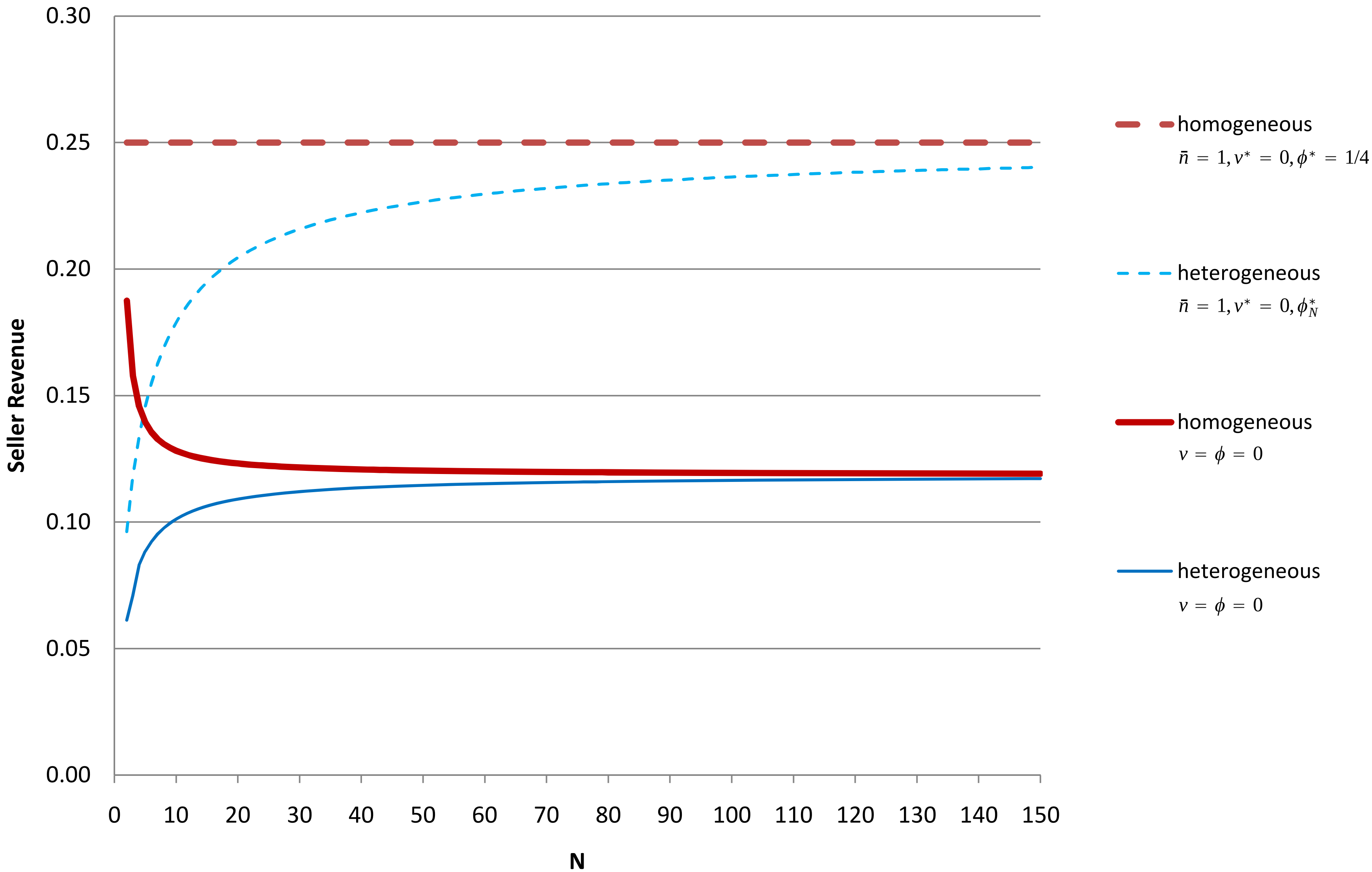

Figure 1: Seller Revenue and the number of bidders

$V_i \sim U[0,1],\ c = 1/4,\ Z_i \sim U[1/4, 1/2]$