A Note on Parameters in Binomial Option Pricing

Garry de Jager

ISSN: 1036-7373
A NOTE ON PARAMETERS IN BINOMIAL OPTION PRICING

Garry de Jager *

November 1990

* School of Finance
Queensland University of Technology
2 George Street
Brisbane Qld 4000
Australia
Fax 61-7-229-8920
Abstract

This paper outlines some difficulties, including arbitrage possibilities, with published binomial option pricing parameters. A new set of parameters are developed, and tested under exacting conditions against the more popular choices.
1.0 INTRODUCTION

Option pricing using the binomial method, first suggested by Sharpe, and popularized by Cox, Ross and Rubinstein [1], requires the use of three parameters:

the size of the geometric upward movement (U),
the downward movement (D), and
the "pseudo probability" of an upward movement (p).

The values allocated to these parameters varies in the literature. The major published sets of parameters ensure that the binomial pricing process converges to the Black-Scholes [2] model for a European option as the number of binomial steps increases and approaches infinity. Values for p, U and D effectively provide a binomial expansion of the stock price at maturity for the risk-neutral world.

However, the 'risk-neutral' argument developed by Cox and Ross [3] shows that the Black-Scholes option price is equivalent to discounting option payoffs at maturity, where these payoffs are based on a lognormal distribution of the stock price with 'drift' of the risk-free rate. Hence, chosen binomial parameters must also achieve a stock price expansion at maturity that converges to this lognormal distribution.

To date, published binomial parameter sets have not ensured equivalence in the appropriate moments of the relevant processes, and this can cause major discrepancies under exacting tests.

Section 2 of this paper presents the three popular binomial parameters sets. Exacting tests are applied in Section 3 and arbitrage possibilities examined in Section 4. Section 5 develops a set of "exact parameters" and Section 6 discusses their application.

2.0 PUBLISHED PARAMETERS

Sets of binomial parameters each require a given volatility, a time to maturity 't', and the number of binomial steps 'n'. The first of the three popular sets is due to Jarrow and Rudd [4]:

\[ U = e^{-\frac{\sigma^2}{2} t + \sigma \sqrt{t}} \]

\[ D = e^{-\sigma \sqrt{t}} \]

\[ p = \frac{1}{2} \]

\[ (1) \]

\[ \frac{(r - \frac{\sigma^2}{2}) t + \sigma \sqrt{t}}{\sqrt{n}} \]
Cox, Ross and Rubinstein [1] developed the following parameters:

\[ U = e^{\frac{r}{\sqrt{n}}} \]

\[ D = U e^{-2\sigma \sqrt{\frac{t}{n}}} \]

\[ p = \frac{1}{2} + \frac{1}{2} \left( \frac{r - \sigma}{\sigma} \right) \sqrt{\frac{t}{n}} \]

Finally there is a set based on Cox, Ross and Rubinstein, often used by practitioners:

\[ U = e^{\frac{r}{\sqrt{n}}} \]

\[ D = U e^{-2\sigma \sqrt{\frac{t}{n}}} \]

\[ p = \frac{R - D}{U - D} \]

3.0 EXACTING TESTS ON THE PARAMETERS

Four exacting tests are employed in pricing European equity calls to establish that there is room for a more precise set of parameters. The distinguishing features are:

i) Zero strike
ii) High volatility
iii) High risk-free rate
iv) Very long dated maturity

3.1 Zero Strike

A zero strike call must have the same price as the stock. In the binomial pricing of a zero strike call, the formula collapses to
In the case of the Jarrow and Rudd parameters, this is equal to
\[ Se^{-rT}[(pU+(1-p)D)]^n \]
\[ Se^{-\frac{\sigma^2 T}{2}} \cosh(\sigma \sqrt{\frac{T}{n}}) \]

which, it can be demonstrated, is always less than S, for finite 'n'. [See Note 1].

All the 'n+1' outcomes at maturity are used in the calculations. The volatility of these outcomes can be measured, and should equal the volatility that was input into the process.

Pricing results of a zero strike call with fairly typical inputs are listed in Table 1 for the three parameter sets.

<table>
<thead>
<tr>
<th>Parameter Set</th>
<th>Call Price</th>
<th>Process Volatility</th>
<th>Upward Mvt Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarrow &amp; Rudd</td>
<td>$99.993</td>
<td>30.00%</td>
<td>0.5</td>
</tr>
<tr>
<td>Cox, Ross &amp; Rub’n &quot;Practitioners&quot;</td>
<td>$99.9976</td>
<td>30.00%</td>
<td>0.5058</td>
</tr>
<tr>
<td>$100</td>
<td>$100</td>
<td>30.00%</td>
<td>0.5058</td>
</tr>
</tbody>
</table>

Stock Price = $100; Maturity = 1 year; Vol = 30%; Risk Free Rate = 8% continuous; 100 binomial steps;

Table 1

The differences from the desired result are not great, but signal problems if harsher inputs are used.

3.2 High Volatility

Extremely high volatility increases the size of the option outcomes at maturity. The Jarrow & Rudd, and the Cox, Ross & Rubinstein parameters rely on large ‘n’ and relatively small volatility for matching of the first moments. At the high volatility used in Table 2, the results are dramatic:

<table>
<thead>
<tr>
<th>Parameter Set</th>
<th>Call Price</th>
<th>Process Volatility</th>
<th>Upward Mvt Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarrow &amp; Rudd</td>
<td>$22.72</td>
<td>400%</td>
<td>0.5</td>
</tr>
<tr>
<td>Cox, Ross &amp; Rub’n &quot;Practitioners&quot;</td>
<td>$40.81</td>
<td>367%</td>
<td>0.301</td>
</tr>
<tr>
<td>$99.48</td>
<td>$99.48</td>
<td>370%</td>
<td>0.310</td>
</tr>
</tbody>
</table>

Stock Price = $100; Strike = $100; Maturity = 2 years; Volatility = 400%; Risk Free Rate = 4% continuous; 50 binomial steps; Black-Scholes Price = $99.55;

Table 2
It may appear, at first, that an error has been made. However, a check indicates that if the strike of this option were altered to zero, the Jarrow & Rudd call price would only increase to $23.17. The above $77.28 discrepancy is purely due to the lack of convergence of the first moments of the two processes.

3.3 High Risk-Free Rate

A major problem with both the Cox, Ross & Rubinstein, and the "Practitioners" parameters, is that they allow probabilities of an upward movement of greater than 1, and hence negative probabilities for downward movements. This has a peculiar effect in the reversing process. If the reversing process is not employed, then every second term in the binomial summation is negative. These effects are seen in Table 3:

<table>
<thead>
<tr>
<th>Parameter Set</th>
<th>Call Price</th>
<th>Process Volatility</th>
<th>Upward Mvt Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarrow &amp; Rudd Cox, Ross &amp; Rub'n &quot;Practitioners&quot;</td>
<td>$79.81</td>
<td>10%</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>$618.12</td>
<td>20,962%</td>
<td>1.295</td>
</tr>
<tr>
<td></td>
<td>$2,140.08</td>
<td>108,500%</td>
<td>1.308</td>
</tr>
</tbody>
</table>

Stock Price = $100; Strike = $100; Maturity = 2 years; Volatility = 10%; Risk Free Rate = 80% continuous; 50 binomial steps; Black-Scholes Price = $79.81;

Table 3

3.4 Very Long Dated Options

Practitioners usually increase the number of binomial steps in relation to the time to maturity. This is not so important if theoretically correct parameters are used. Where there is not accurate matching of the first moment, an increase in time to maturity exacerbates inaccuracies as the example in Table 4 demonstrates for 5, 20 and 100 year options.

<table>
<thead>
<tr>
<th>Parameter Set</th>
<th>5 Years</th>
<th>20 Years</th>
<th>100 Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black-Scholes</td>
<td>$42.01</td>
<td>$82.77</td>
<td>$100.00</td>
</tr>
<tr>
<td>Jarrow &amp; Rudd</td>
<td>$41.94</td>
<td>$82.23</td>
<td>$61.06</td>
</tr>
<tr>
<td>Cox, Ross &amp; Rub'n &quot;Practitioners&quot;</td>
<td>$41.08</td>
<td>$80.88</td>
<td>$20.39</td>
</tr>
<tr>
<td></td>
<td>$41.89</td>
<td>$82.64</td>
<td>$100.00</td>
</tr>
</tbody>
</table>

Stock Price = $100; Strike = $100; Vol = 30%; Risk Free Rate = 8% continuous; 50 binomial steps;

Table 4

4.0 ARBITRAGE

In theory, arbitrage opportunities exist for all the above parameter sets which
is now illustrated using the Jarrow and Rudd values for U, D and p, in the
binomial pricing process. We calculate the value of a call where there is just
one of the 'n' binomial periods still to run i.e. there is 't/n' to maturity.
Assume the option is in-the-money, the stock price is S, the continuous interest
rate to maturity is 'r', and the call strike is K. The call price with 't/n' to run
is given by

$$e^{-rt} \left[p(US-K)+(1-p)(DS-K)\right]$$

and the hedge ratio will be 1.

The creation of the 'risk-free' portfolio consisting of one short stock and one
long call will generate cash, which if invested at the risk-free rate, will yield at
maturity

$$S[e^{-rt}U-(1-p)D] + K$$

At maturity, the holder of this portfolio delivers the short stock by virtue of
exercising the call option which costs K, i.e. the net amount garnered will be

$$S[e^{-rt}U-(1-p)D]$$

For the Jarrow & Rudd parameters, this amount is

$$Se^{-rt} \left[1-e^{-\frac{t}{n}} \cosh(\sqrt{\frac{1}{n}})\right]$$

which is always positive and is approximately equal to

$$S\frac{\sigma^2t^2}{4n^2}$$

Thus there is a small, but certain, arbitrage profit. As is demonstrated in the
next section, the causes of both the arbitrage problem and the valuation of the
options in Tables 1-4 lies with the solving of inappropriate equations for the
binomial parameters.

5.0 DEVELOPING BINOMIAL PARAMETERS

Although most of the discrepancies examined are minor if enough binomial
steps are employed, a set of parameters that do not give rise to these
discrepancies should be determined.

The option pricing formula developed by Black-Scholes is derived using a
process for the stock that is lognormal, or Brownian motion. The hedging
portfolio in both the lognormal process and binomial process was found to be
risk free.
Cox and Ross [2] pointed out that since option pricing is independent of the expected return on the stock, a "risk neutral world" approach to pricing options can be postulated i.e. discounting the expected payoffs of the option at maturity, at the risk free rate.

This simplifying assumption allows the binomial parameters to be chosen so that at maturity the binomial process for the stock price-converges to the lognormal process. The payoffs are then discounted at the risk free rate in both cases and the binomial result should equal that provided by the Black-Scholes formula.

The DeMoivre-Laplace theorem, dated 1714, states that the Bernouilli (binomial) process converges to the normal process if their first two moments match. An elegant proof is contained in Rozanov [5].

Option pricing depends on the binomial converging to the log-normal process, and a similar proof to that supplied by Rozanov proves the convergence, if the first two moments (of the lognormal and binomial expansion of the stock price) are equivalent.

Equations for matching the moments of the normal distribution and the binomial expansion of the logs of the price relatives are:

\[(4a) \quad n[p\ln(U)+(1-p)\ln(D)]-rt-\frac{\sigma^2 t}{2}\]

\[(4b) \quad np(1-p)[\ln(U)-\ln(D)]^2-\sigma^2 t\]

with the binomial moments on the left. Another set of equations can be developed to match the moments of the lognormal process with the binomial expansion of the price relatives themselves:

\[(5a) \quad [pU+(1-p)D]^n-e^r\]

\[(5b) \quad [pU^2+(1-p)D^2]^n-[pU+(1-p)D]^{2n}-e^{2r}[e^{\sigma^2}-1]\]

The Jarrow and Rudd parameters in equations (1) are derived from equations (4a) and (4b). Although there are three unknowns and only two equations, the degree of freedom is removed by setting \(p = 0.5\), which ensures a coefficient of skew of zero i.e. ensures equivalence in the third moment as well as the first two.

To formulate a set of "exact parameters", a solution is required for a "hybrid" set of equations i.e. (5a) and (4b). Equation (5a) is required so that the expected mean of the binomial process accurately matches the expected mean of the lognormal process --- ensuring that the zero strike call always equals the stock price.

Equation (4b) is needed so that the volatility (i.e. standard deviations of the logs of the returns) of the binomial process is exactly that used in the lognormal process, since it is this volatility that is a direct input into the Black-
Scholes formula.

From a theoretical point of view, it is important to use the logs of the price relatives in calculating the volatility, since only they provide linearity over time with respect to the variance. For a lognormal process it is not accurate to say that the square of the standard deviation of returns over 1 year is equal to 12 times the square of the monthly standard deviation of returns [see Note 2.]

There are two equations with three unknowns. Setting \( p = 0.5 \) to avoid skew, we have two equations with two unknowns, yielding the following parameters:

\[
\begin{align*}
U &= \frac{r^n}{2e^{n \sigma}} - \frac{\sigma^2}{1 + e^{n \sigma}} \\
D &= -ue^{-\sigma^2/n} \\
p &= \frac{1}{2}
\end{align*}
\]

(6) — — —

The formula for the upward movement can be rewritten:

\[
\begin{align*}
U &= \frac{r^n}{e^{n \sigma}} \frac{\sqrt{n}}{\cosh(\sqrt{n} \sigma)}
\end{align*}
\]

(7) — — —

An unpublished paper by Heath, Jarrow & Morton [6] on interest rate options uses a related set of binomial parameters for its evolution of the entire yield curve, but does not generalize the result for standard option pricing.

All the problems mentioned heretofore disappear with this set of parameters:

i) The zero strike call in Table 1 is valued at $100 exactly.

ii) The high volatility call in Table 2 is valued at $99.08 high is very close to the Black-Scholes value.

iii) The high risk-free rate call in Table 3 is valued at $79.81, matching the Black-Scholes value.

iv) Long dated option problems presented in Table 4 largely evaporate with prices matching the Black-Scholes values even though only 50 binomial steps are employed. Results were within .01% as the following prices show:
<table>
<thead>
<tr>
<th>Maturity</th>
<th>Black_Scholes</th>
<th>Binomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 years</td>
<td>$42.01</td>
<td>$41.97</td>
</tr>
<tr>
<td>20 years</td>
<td>$82.77</td>
<td>$82.75</td>
</tr>
<tr>
<td>100 years</td>
<td>$100.00</td>
<td>$100.00</td>
</tr>
</tbody>
</table>

This suggests fewer binomial steps may need to be employed than the conventional wisdom demands for longer dated options.

v) The arbitrage problem developed in Section 4 disappears with the new parameters.

Note that it is quite possible to match the first two moments without regard to the third moment and a sufficiently large number of iterations will ensure higher moments also converge. Thus, theoretically, there are an infinite number of parameters sets, one of which is in equations (8).

\[
U = \frac{e^{\frac{rt}{n}}}{.45+.55e^{-200\sqrt{\frac{t}{99n}}}}
\]

\[
D = \frac{e^{-\frac{rt}{n}}}{.55+.45e^{+200\sqrt{\frac{t}{99n}}}}
\]

\[
p = \frac{9}{20}
\]

and these parameters also provide good prices in each of the four worrisome cases.

6.0 GENERAL APPLICATIONS

6.1 American Options

Pricing of American options can proceed simply by substituting the parameters developed in (6) for those used by the popular algorithms. Fewer binomial steps may be needed in some instances.

6.2 Options on Futures

For equivalence to the Black [7] commodity option formula, the parameters in equations (6) are modified, setting \( r = 0 \), and the volatility becomes the volatility of the futures price, not the spot.

6.3 Monte Carlo Simulations

In the Monte Carlo approach to pricing, suggested by Boyle [8], the time to maturity is divided into 'n' periods and a "drift" for each period is calculated. Rather than the standard drift of
the appropriate drift becomes

\[ \frac{\frac{r}{n}}{e^{\frac{\sigma^2 T}{n}}} \cosh(\sigma \sqrt{\frac{T}{n}}) \]

As before, the variable multiplier for each period is

\[ e^{z \sqrt{\frac{T}{n}}} \]

where 'z' is the normally distributed random variable.

6.4 Foreign Exchange Options

Equivalence to the foreign exchange option formula developed by Grabbe [9], is supplied using the parameters in (6) with the usual modification that 'r' is replaced by 'rd-rf' where rd and rf are the domestic and foreign continuous rates respectively.

6.5 Exotic Options

Specialized options requiring numerical solutions usually depend on a set of binomial parameters. These options could use the parameters specified in (6).

7.0 CONCLUSIONS

The published parameters generally experience difficulties only under extreme conditions.

Notwithstanding the practical methods of overcoming the problems, the new set of parameters are more theoretically accurate, provide greater degrees of comfort as tools to cope with all situations, and offer the prospect of fewer binomial steps for long dated options.
1. The binomial formula collapses to:

\[ Se^{-n[pU+(1-p)D]} \]

On substitution of the Jarrow & Rudd values, the call equals

\[ Se^{-\frac{\sigma^2 t}{2}} \left( e^{\frac{\sigma \sqrt{t}}{n}} + e^{-\frac{\sigma \sqrt{t}}{n}} \right) \]

which can be expanded to

\[ Se^{-\frac{\sigma^4 t}{4}} \left[ 1 + \frac{\sigma^2 t}{2} + \frac{\sigma^4 t^2}{4!n^2} \right] \]

and for positive volatility, 'n' and 't' is always less than

\[ Se^{-\frac{\sigma^2 t}{2}} e^{\frac{\sigma^4 t}{4}} \]

i.e. less than the stock price

2. Using standard calculus, it can be shown that for a lognormal distribution with an annual volatility (standard deviation of the logs of the price relatives), the variance (of the logs of the price relatives) over 't' years is given by

\[ \sigma^2 t \]

hence the annual variance is

\[ \sigma^2 \]

and the monthly variance is

\[ \frac{\sigma^2}{12} \]

i.e. there is linearity. However, for the same lognormal distribution, the variance of the price relatives themselves, for 't' years, is given by

\[ e^{2\sigma^2 t}[e^{\sigma^2} - 1] \]

Thus for one year, the variance is

\[ e^{2\sigma^2}[e^{\sigma^2} - 1] \]
and for one month, the variance is
\[
\frac{\tau \sigma^2}{e^{\frac{\sigma^2}{12}}[e^{\frac{\sigma^2}{12}} - 1]}
\]
and the latter is not exactly one twelfth of the former, although for small variances and shorter periods, that relationship is a good approximation (and popularly used to calculate stock betas).
REFERENCES


