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A Dynamic Analysis of Cournot Duopoly in Imperfectly Competitive Product and Factor Markets

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Abstract

Assuming identical firms, a linear market demand function and a single factor of production, labour, we analyse the existence and stability of a homogeneous Cournot duopoly facing imperfect competition in both product and factor markets. We establish the possibility of antisymmetric equilibria separated by a unique symmetric equilibrium. Under certain assumptions on the wage function we are able to demonstrate local stability of the unique symmetric equilibrium, and its global stability under the assumption of a convex reaction function.

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1. Introduction

Since Theocharis (1959), many economists have analysed the stability of the Cournot oligopoly equilibrium under various assumptions. A summary of the main results up to 1976 has been given by Okuguchi (1986) and up to 1990 by Okuguchi and Szidarovszky (1990). All these results, however, have been obtained on the basis of cost functions for oligopoloitic firms assumed to be facing perfect competition in factor markets. The cost function approach is appropriate only if firms engage in productive activities in perfectly competitive factor markets. Okuguchi (1993, 1995) was the first to formulate the Cournot Oligopoly model where product and factor markets are simultaneously imperfectly competitive and established the existence of a unique Cournot oligopoly - oligopsony equilibrium without recourse to the cost function approach.

In this paper we will examine the stability properties of the equilibrium for his model after preliminary analysis of its existence. In so doing we will assume that the number of firms is two and that there is one factor. We will furthermore assume that the firms are symmetric. Initially it might appear that our symmetric Cournot duopoly - duopsony has only a symmetric equilibrium but we will show that this is not the case. Indeed our model may have a symmetric equilibrium as well as multiple asymmetric ones. Novshek (1984), Okuguchi (1984) and De Finaja (1994) have pointed out the possibility of existence of asymmetric equilibria for the symmetric Cournot oligopoly where all firms have an identical cost function. Okuguchi and Chiarella (1994) have analysed the stability properties of asymmetric equilibria for symmetric Cournot oligopoly in perfectly competitive factor markets.

2. The Model

We adopt the framework of Okuguchi (1993) with the special case of identical firms, linear market demand function, and most importantly, we assume a single factor of production, namely labour.

Thus each firm has a production function given by

$$x_i = L_i^{\beta}, \qquad 0 < \beta < 1, \qquad i = 1, 2, \tag{1}$$

where x_i and L_i are respectively output of, and labour employed by, firm i. Setting

$$Q \equiv x_1 + x_2, \tag{2}$$

we assume a linear inverse demand function which we write as

$$p(Q) = a - bQ \qquad (a > 0,\ b > 0). \tag{3}$$

We assume the wage rate is given by $w(L), \quad w'(L) > 0$ where $L \equiv L_1 + L_2$ and at subsequent points in our discussion we will invoke one or other of the following two assumptions about the wage function:-

Assumption W1

$$w'(L) + L_i w''(L) \geq 0 \qquad for\ i = 1, 2.$$

This assumption states that the marginal labour cost of each firm is a non-decreasing function of the labour input of the rival firm.

Assumption W2

$$w(L) = \gamma L^{\theta}, \qquad \gamma > 0, \qquad \theta \geq 1.$$

We observe that assumption W1 is satisfied by this flexible functional form which imposes a convex wage function.

The profit function of firm i is given by

$$\begin{aligned} \pi_i &= [a - b(x_1 + x_2)]x_i - w(L)L_i, \\ &= [a - b(L_1^{\beta} + L_2^{\beta})]L_i^{\beta} - w(L_1 + L_2)L_i, \qquad (i = \end{aligned} \tag{4}$$

Each firm seeks L_i so as to maximise π_i. However, since there is a 1-1 correspondence between L_i and x_i we can equally allow x_i to be the decision variable. We follow this latter choice as it simplifies slightly the algebra in the ensuing analysis. Thus in terms of output, the profit of firm i may be written

$$\pi_i \equiv [a - b(x_1 + x_2)]x_i - x_i^{\frac{1}{\beta}} w(x_1^{\frac{1}{\beta}} + x_2^{\frac{1}{\beta}}), \qquad (i = \tag{5}$$

We note that

$$\frac{\partial \pi_i}{\partial x_i} = a - b(2x_i + x_j) - \frac{1}{\beta} x_i^{\frac{1}{\beta}-1} [w(x_1^{\frac{1}{\beta}} + x_2^{\frac{1}{\beta}}) + x_i^{\frac{1}{\beta}} w'(x_1^{\frac{1}{\beta}} + x \tag{6}$$

3. Reaction Functions and Equilibrium Analysis

Setting marginal profit of firm 1 to zero we obtain the equation that determines firm 1's reaction function viz.

$$a - 2bx_1 = bx_2 + \frac{1}{\beta} x_1^{\frac{1}{\beta}-1} [w(x_1^{\frac{1}{\beta}} + x_2^{\frac{1}{\beta}}) + x_1^{\frac{1}{\beta}} w'(x_1^{\frac{1}{\beta}} + x_2^{\frac{1}{\beta}}) \qquad \textbf{(7)}$$

Define the function

$$g(x,y) \equiv by + \frac{1}{\beta} x^{\frac{1}{\beta}-1} \left[w(x^{\frac{1}{\beta}} + y^{\frac{1}{\beta}}) + x^{\frac{1}{\beta}} w'(x^{\frac{1}{\beta}} + y^{\frac{1}{\beta}}) \right] \qquad \textbf{(8)}$$

Then equation (7) may be written

$$a - 2bx_1 = g(x_1, x_2). \qquad \textbf{(9a)}$$

Similarly firms 2's profit maximising condition may be rewritten

$$a - 2bx_2 = g(x_2, x_1). \qquad \textbf{(9b)}$$

Equations (9) determine each firm's reaction function, viz

$$x_1 = G(x_2), \qquad \textbf{(10a)}$$

$$x_2 = G(x_1). \qquad \textbf{(10b)}$$

We seek to determine the slope and convexity/concavity of the function G. The main properties of this function are stated as:

Proposition 1: *Under assumption W1 the function G has the properties*

(i) $G' < 0$,
(ii) $G(0) < a/2b$,
(iii) $G(a/b) = 0$.

Proof:

From equation (8) we may calculate the partial derivatives (using subscripts 1, 2 to denote partial derivatives with respect to first and second arguments)

$$g_1(x,y) = \frac{1}{\beta}\left(\frac{1}{\beta}-1\right)x^{\frac{1}{\beta}-2}[w(.)+x^{\frac{1}{\beta}}w'(.)]$$

$$+\frac{1}{\beta^2}x^{2(\frac{1}{\beta}-1)}[2w'(.)+x^{\frac{1}{\beta}}w''(.)], \qquad \textbf{(11a)}$$

$$g_2(x,y) = b+\frac{1}{\beta^2}(xy)^{\frac{1}{\beta}-1}[w'(.)+x^{\frac{1}{\beta}}w''(.)], \qquad \textbf{(11b)}$$

We note that under assumption W1 we may assert

$$g_1(x,y) \geq 0, \qquad g_2(x,y) \geq 0, \qquad \textbf{(12)}$$

for all $x \geq 0,\ y \geq 0$ ·

Consider the equation

$$a - 2by = g(y,x) \qquad \textbf{(13)}$$

which defines the reaction function

$$y = G(x). \qquad \textbf{(14)}$$

Differentiating (13) implicitly and rearranging we find

$$\frac{dy}{dx} = \frac{-g_2(y,x)}{(2b + g_1(y,x))} \qquad \textbf{(15)}$$

which in light of (12) implies that G' < 0 under assumption W1 i.e. the reaction function is downward sloping over its domain.

We then note that G(0) is given by the solution of

$$a - 2by = g(y,0) = \frac{1}{\beta}y^{\frac{1}{\beta}-1}[w(y^{\frac{1}{\beta}}) + y^{\frac{1}{\beta}}w'(y^{\frac{1}{\beta}})].$$

By assumption W1 and the further assumption on $w'(.)$ that

$$\lim_{y \to 0} y^{\frac{2}{\beta}-1} w'(y^{\frac{1}{\beta}}) = 0,$$

the function g(y, 0) is increasing from the origin. Hence G(0) must satisfy

$$G(0) < \frac{a}{2b}.$$

The value of x such that G(x) = 0 is given by the solution of

$$a - 2b.0 = g(0,x) = bx.$$

Hence $G\left(\frac{a}{b}\right) = 0.$ ■

In order to investigate the concavity/convexity of G we calculate the second derivatives

$$g_{11}(x,y) = \frac{1}{\beta}\left(\frac{1}{\beta}-1\right)\left(\frac{1}{\beta}-2\right)x^{\frac{1}{\beta}-3}[w(.)+x^{\frac{1}{\beta}}w'(.)]$$

$$+\frac{3}{\beta^2}\left(\frac{1}{\beta}-1\right)x^{\frac{2}{\beta}-3}[2w'(.)+x^{\frac{1}{\beta}}w''(.)]$$

$$+\frac{1}{\beta^3}x^{\frac{3}{\beta}-3}[3w''(.)+x^{\frac{1}{\beta}}w'''(.)],$$

$$g_{22}(x,y) = \frac{1}{\beta^2}\left(\frac{1}{\beta}-1\right)x^{\frac{1}{\beta}-1}y^{\frac{1}{\beta}-2}[w'(.)+x^{\frac{1}{\beta}}w''(.)]$$

$$+\frac{1}{\beta^3}x^{\frac{1}{\beta}-1}y^{\frac{2}{\beta}-2}[w''(.)+x^{\frac{1}{\beta}}w'''(.)],$$

$$g_{12}(x,y) = \frac{1}{\beta^2}\left(\frac{1}{\beta}-1\right)x^{\frac{1}{\beta}-2}y^{\frac{1}{\beta}-1}\left[w'(.)+x^{\frac{1}{\beta}}w''(.)\right]$$

$$+\frac{1}{\beta^3}x^{\frac{2}{\beta}-2}y^{\frac{1}{\beta}-1}\left[2w''(.)+x^{\frac{1}{\beta}}w'''(.)\right].$$

We note that under assumption W2 we may assert

$$g_{22}(x,y) \geq 0, \qquad g_{12}(x,y) \geq 0, \tag{16a}$$

for all $x \geq 0,\ y \geq 0$ ·

Furthermore

$$g_{11}(x,y) \geq 0, \tag{16b}$$

under assumption W2 provided $\beta \leq \frac{1}{2}$. However for $\frac{1}{2} < \beta \leq 1$, $g_{11}(x,y)$ is difficult to sign in general.

Differentiating implicitly once again we obtain

$$-(2b+g_1)\frac{d^2y}{dx^2} = g_{11}\left(\frac{dy}{dx}\right)^2 + 2g_{12}\frac{dy}{dx} + g_{22}. \tag{17}$$

The coefficient of $\frac{d^2y}{dx^2}$ on the LHS is clearly negative, however from equations (13) we see that the term on the RHS is of indeterminate sign (even under assumption W2), so we need to consider the possibilities of $G'' > 0$, $G'' < 0$ and G'' being of both signs over its domain.

Hence we can sketch in figure 1 all the possible configurations for the reaction function. The cases in which either $G'' > 0$ or $G'' < 0$ clearly from geometrical considerations lead to a unique equilibrium point. However the case in which G'' is both positive and negative over its domain raises the possibility of multiple equilibria.

Equilibrium points are given by the simultaneous solution of the equations

$$y = G(x),$$

$$x = G(y).$$

Clearly *symmetric equilibria* (i.e. $\bar{x} = \bar{y}$) may exist. On the other hand if the pair

$(\bar{x}, \bar{y})$ $(\bar{y} \neq \bar{x})$ is an equilibrium solution then so is the pair $(\bar{y}, \bar{x})$. Such equilibria are *antisymmetric* and occur in pairs. No other type of equilibrium is possible.

Solutions for y satisfy

$$y = G(G(y)) \equiv H(y). \qquad \textbf{(18)}$$

We observe that

$$H'(y) = G'(G(y))G'(y) > 0. \qquad \textbf{(19)}$$

We note also that

$$H(0) = G(G(0)) > 0, \qquad \textbf{(20a)}$$

and that

$$H\left(\frac{a}{b}\right) = G\left(G\left(\frac{a}{b}\right)\right) = G(0) < \frac{a}{2b} < \frac{a}{b}. \qquad \textbf{(20b)}$$

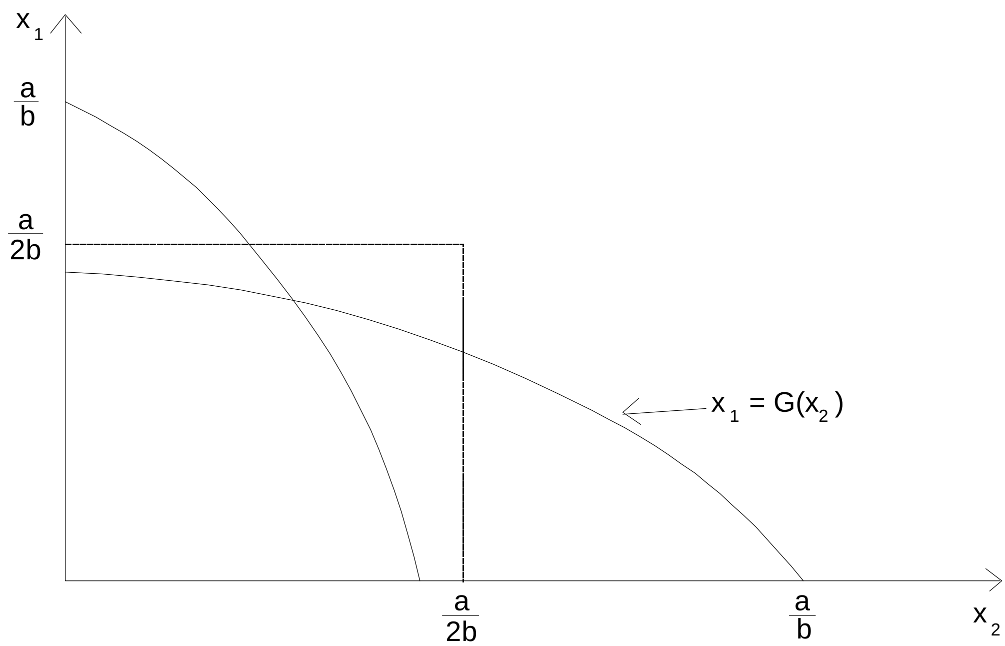

Figure 1(a) G" < 0

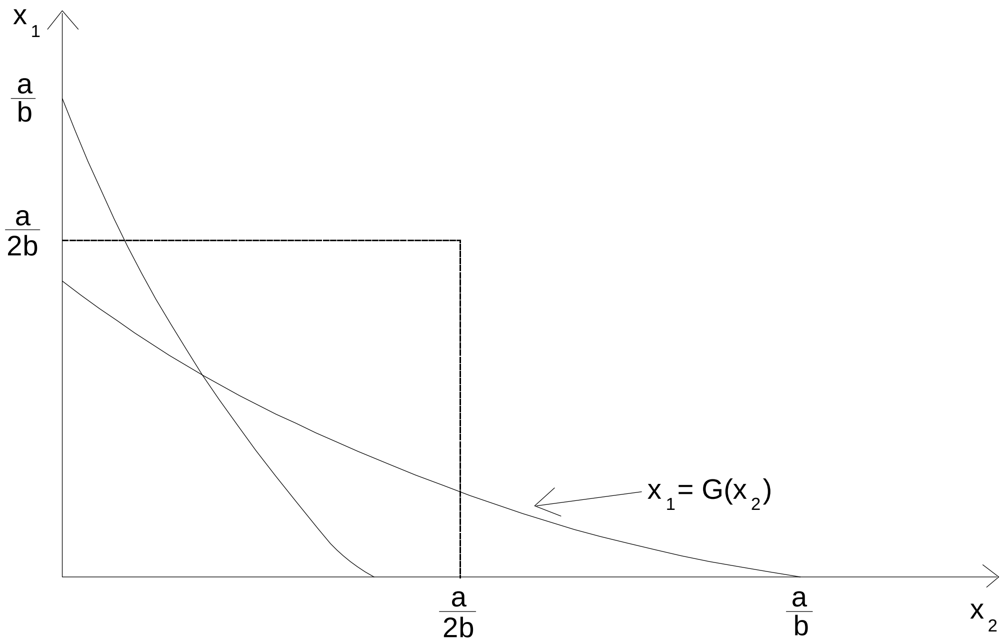

Figure 1(b) G" > 0

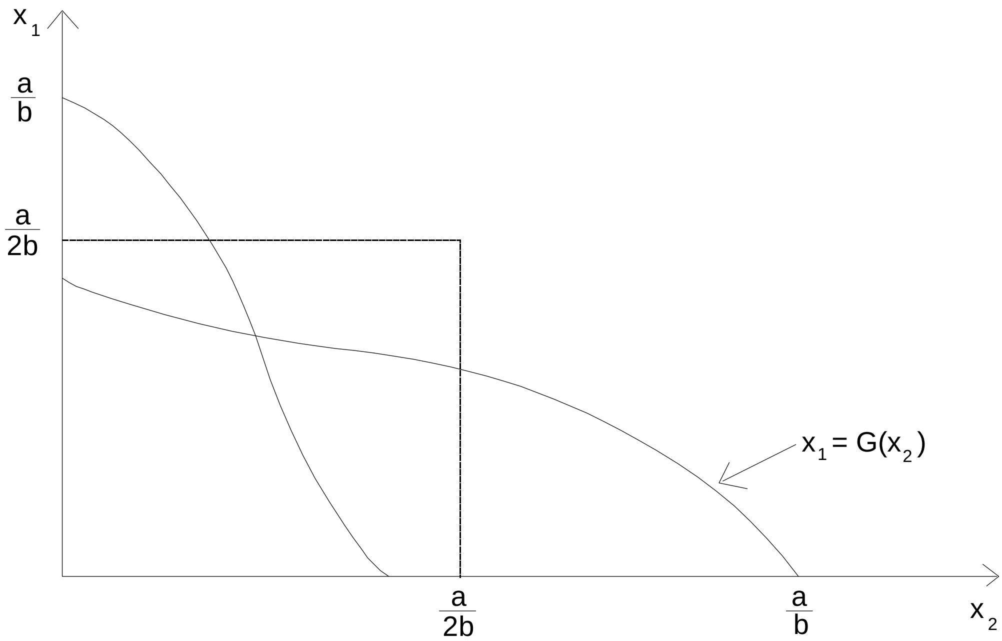

Figure 1(c) G" > 0 and < 0 over its domain

It follows that H(y) intersects y at least once from above as shown in figure 2. However multiple equilibria remain a possibility at this stage since we are not able to tie down the sign of H'' without further restrictions.

We first prove a proposition about symmetric equilibria:

Proposition 2: *There is a unique symmetric equilibrium ($\bar{x}, \bar{x}$) which is characterised by the property*

$$0 < H^{/}(\bar{x}) < 1.$$

Proof

We note that a symmetric equilibrium point is determined by

$$\bar{x}_1 = H(\bar{x}_1), \qquad \bar{x}_2 = H(\bar{x}_2).$$

We set

$$\bar{x}_1 = \bar{x}_2 \equiv \bar{x}. \tag{21}$$

Let $w^{/}(.)$, $w^{//}(.)$ represent the first and second derivatives of the wage function evaluated at the equilibrium point.

From equation (15) we can (after a slight rearrangement) write

$$G^{/}(\bar{x}) = \frac{-\rho}{\rho + \delta} \tag{22}$$

where

$$\rho \equiv b + \frac{1}{\beta^2}\bar{x}^{2\left(\frac{1}{\beta}-1\right)}[w^{/}(.) + \bar{x}^{\frac{1}{\beta}}w^{//}(.)], \tag{23a}$$

$$\delta \equiv b + \frac{1}{\beta}\left(\frac{1}{\beta}-1\right)\bar{x}^{\frac{1}{\beta}-2}[w(.) + \bar{x}^{\frac{1}{\beta}}w^{/}(.)] + \frac{1}{\beta^2}\bar{x}^{2\left(\frac{1}{\beta}-\right.} \tag{23b}$$

The assumption $w^{/} > 0$ suffices to guarantee $\delta > 0$, whilst assumption W1 guarantees $\rho > 0$.

It is then a simple matter to deduce that

$$|G^{/}(\bar{x})| < 1 \tag{24}$$

It follows from equation (19) that $H'(\bar{x}) < 1$. It has already been demonstrated that $H'(x) > 0$. If follows that the equilibrium is unique. A consequence of proposition 2 is that at a symmetric equilibrium H(y) cannot intersect the line y from below in figure 2. ■

We note in order to obtain a pair of antisymmetric equilibria there will have to be multiple intersections of graphs of y and H(y). Given the properties of the function H already demonstrated multiple equilibria can only occur in odd numbers (we exclude consideration of tangency intersections which are structurally unstable). Figure 2b shows the case of three equilibrium points. We state the following proposition concerning the disposition of symmetric and anti-symmetric equilibria:

Proposition 3: *Antisymmetric equilibria (if they occur) are separated by the unique symmetric equilibrium.*

Proof Our proof is based on geometric reasoning.

First we observe that the properties of the function G ensure that equilibrium points must lie within the square [0, a/2b] x [0, a/2b]. Secondly observe that antisymmetric equilibria by their definition lie on either side of the 45^0 line within this sequence. It follows that antisymmetric equilibria are separated by the unique symmetric equilibrium. ■

Thus in the case of these equilibrium points as in figure 2b, the pairs (x_1, x_3), (x_3, x_1) are the antisymmetric equilibria and the pair (x_2, x_2) is the symmetric equilibrium. For higher order multiple equilibria (i.e. 5, 7, 9 etc) the middle pair will always be the symmetric equilibrium, as illustrated in figure 2c for the case of 5 equilibria.

We can assert the following proposition concerning multiple equilibria:

Proposition 4: *Multiple equilibria of order 3, 7, 11, ... cannot occur.*

Proof Suppose multiple equilibria of order 3 do occur. Then given the properties of the function H it must be the case that at the (middle) symmetric equilibrium H' > 1. This is so since at the middle equilibrium point the function H(y) intersects the line y from below in the hypothesised situation.

However we have shown in proposition 2 that at a symmetric equilibrium H' < 1. Hence equilibria of order 3 would imply a contradiction and cannot occur.

Similar reasoning applies to equilibria of order 7, 11 etc. ■

It does not seem possible to rule out multiple equilibria of order 5, 9, 13, ... etc without imposing conditions on the function or which are difficult to interpret economically. We leave as an open research question the conjecture that no multiple equilibria of any order can occur under the assumption $w'(L) > 0$. We have not been able to generate any counter examples using graphical computer packages and the assumption W2.

Allowing increasing marginal product of labour would certainly introduce the possibility of multiple equilibria. This also remains a topic of future research.

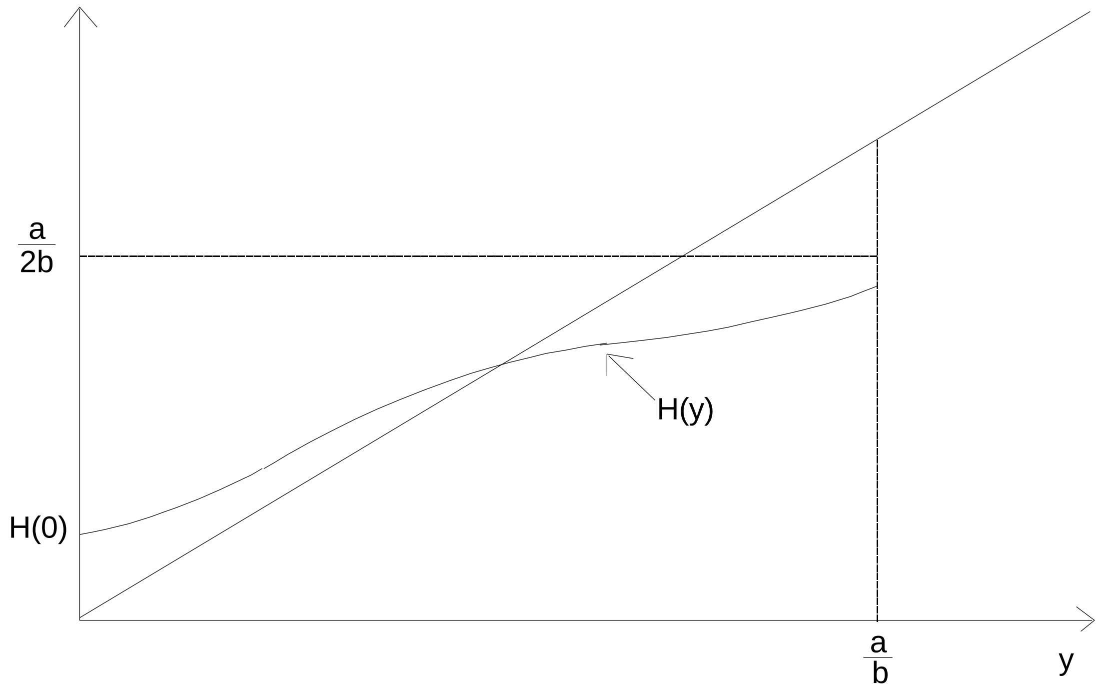

Figure 2(a)

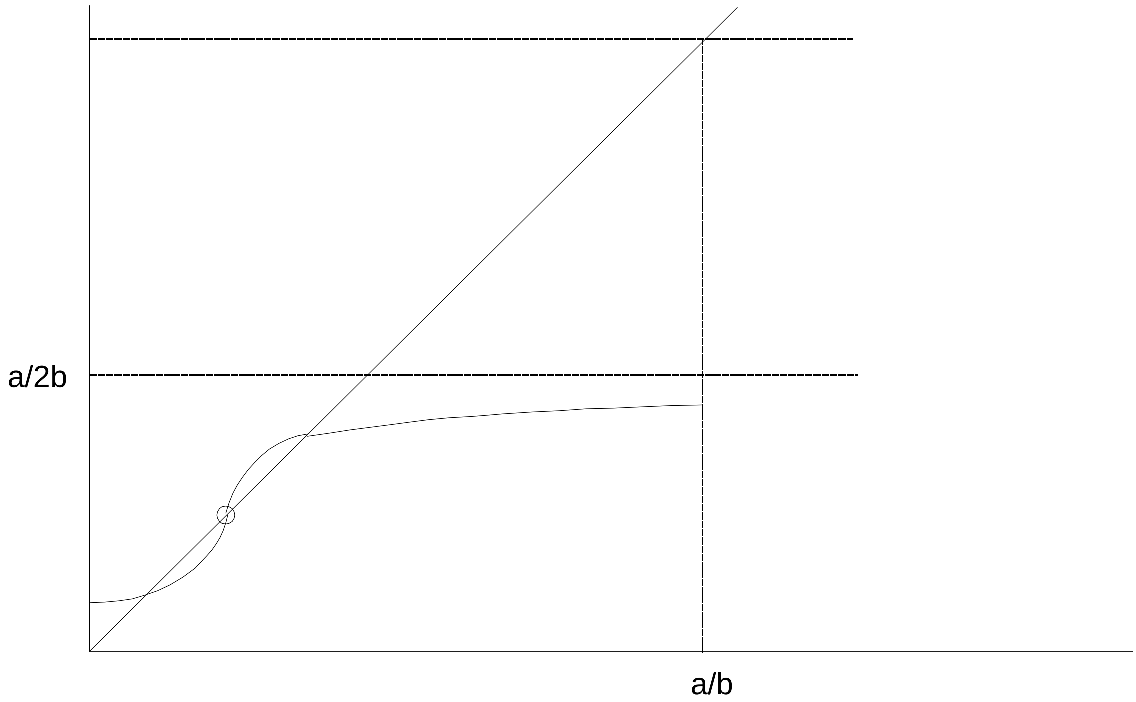

Figure 2(b)

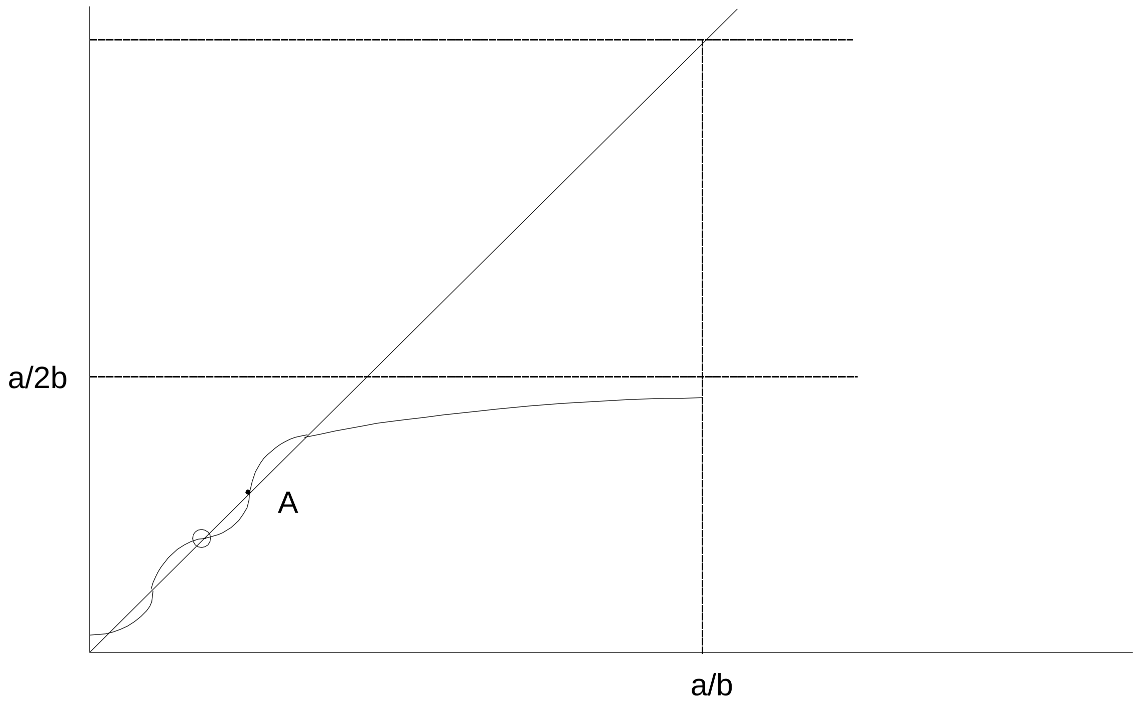

Figure 2(c)

4. Dynamic Adjustment and Stability

We consider the adjustment process whereby each firm adjusts to desired output at some finite speed of adjustment i.e.

$$\dot{x}_1 = k[G(x_2) - x_1], \qquad \textbf{(25a)}$$

$$\dot{x}_2 = k[G(x_1) - x_2], \qquad \textbf{(25b)}$$

where k > 0.

We consider here only the case satisfying;

Assumption W3

The function w(L) satisfies conditions guaranteeing that the function H only intersect the 45^o line once.

Hence we are considering the case of a unique symmetric equilibrium.

In figure 3 we sketch the phase plane of the dynamical system (25), and its vector field. The direction arrows of the vector field suggest global stability of the equilibrium. We are able to prove some precise propositions about stability of the equilibrium.

Proposition 6: ***Under assumption W3 the equilibrium point of the dynamical system (25) is locally asymptotically stable.***

Proof: The Jacobian of the dynamical system (25) at $\bar{x}$ is calculated as

$$\bar{J} = \begin{bmatrix} -k & kG'(\bar{x}) \\ kG'(\bar{x}) & -k \end{bmatrix},$$

and we see that

$$trace(\bar{J}) = -2k < 0,$$

$$\det(\bar{J}) = k^2[1 - [G'(\bar{x})]^2] > 0 \qquad (by\ equation\ 24).$$

Hence the result follows. ■

Proposition 7: ***Under the additional assumption that*** $G''(x) > 0$ ***for 0 < x < b/a, the equilibrium point of the dynamical system (25) is globally asymptotically stable.***

Proof: We note first that

$$G'(0) = -\frac{1}{2},$$

which combined with the assumption $G''(x) > 0$ **and the property** $G'(x) < 0$ **implies**

$$-\frac{1}{2} < G'(x) < 0,$$

for all x $\in$ **(0, b/a).**

The Jacobian of the dynamical system (25) at a general point $(x_1, x_2) \in (0, b/a) \times (0, b/a)$ **is given by**

$$J = \begin{bmatrix} -k & kG'(x_2) \\ kG'(x_1) & -k \end{bmatrix},$$

from which

$$trace(J) = -2k < 0$$

$$\det(J) = k^2[1 - G'(x_1)G'(x_2)] > 0.$$

The result then follows by the Olech (1963) theorem. ■

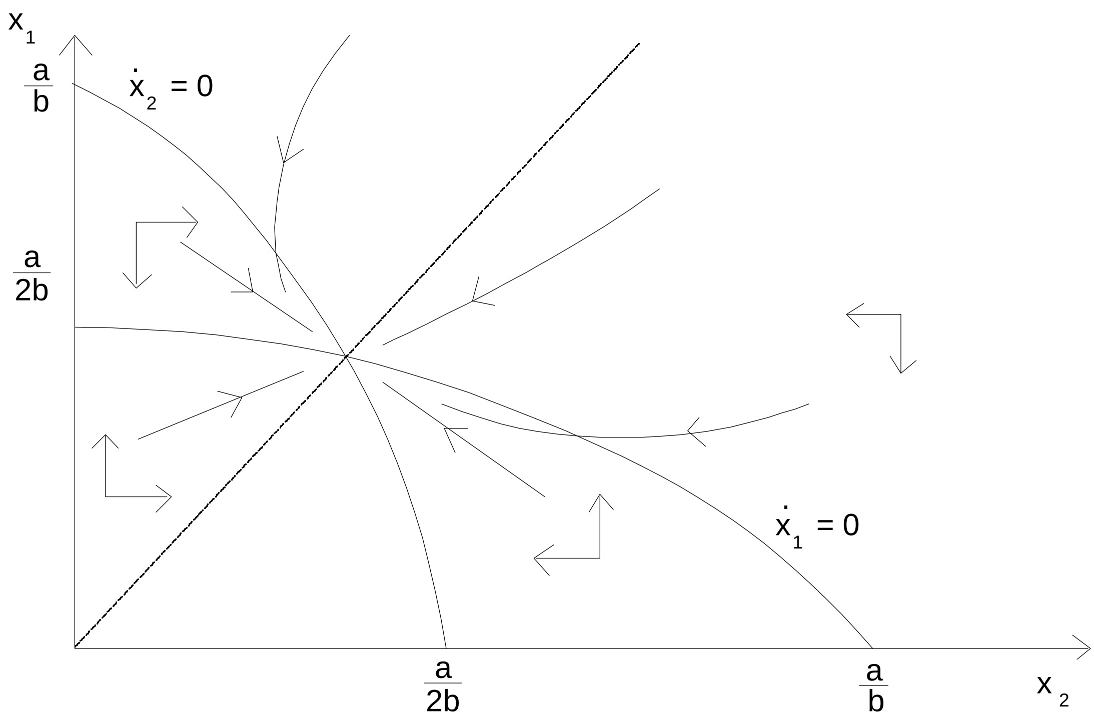

Figure 3

5. Conclusion

In Section 2 we have formulated the Cournot duopoly of identical firms facing imperfect competition in both the product and factor markets. In Section 3 we have established the possibility of the existence of both symmetric and asymmetric equilibria which exhaust the set of possible equilibria. We have further shown that the symmetric equilibrium is unique and will separate asymmetric equilibria if these occur. In Section 4, we have proven the local stability of this unique equilibrium assuming restrictions on the wage function which guarantee a unique symmetric equilibrium. Under the further restriction that the reaction function of each firm is convex we have established global stability of this equilibria.

Our analysis has been conducted using a linear inverse demand function, however we know that it is possible to obtain stability of the Cournot equilibrium under general demand functions containing the linear demand function as a special case, so we feel that this assumption may not be too restrictive. Furthermore, we have assumed away capital as a variable of the production function. In this sense what we have achieved in this paper is quite modest. However, we believe that we have provided a framework and useful reference point for further analysis of the dynamic properties of the Cournot equilibrium in imperfectly competitive product and factor markets. Further research should introduce the second factor of production, capital. It is also important to consider

the effect on stability of the various expectational schemes discussed in Okuguchi (1976). We know from Chiarella and Khomin (1996) in the case of the standard (i.e. firms facing perfect competition in factor markets) Cournot model that this easily leads to the loss of local stability accompanied by Hopf bifurcations which indicate a fairly complex nonlinear dynamic behaviour.

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