Modelling the Yields on Australian Coupon Paying Bonds

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Abstract

Before coupon bond data can be used to make term structure inferences it must be adjusted to account for the coupon effect. This paper compares the performance of two alternative adjustment methods, namely a duration based adjustment of term and a zero coupon method that adjusts yields. The comparison of the two adjustment methods was carried out on unique data set on Australian traded bonds.

The duration adjustment method models coupon bond yields as a function of duration and convexity. The estimation results indicated that despite the apparent attractiveness of high convexity to bond holders that convexity was not a significant determinant of yields. In making the zero coupon adjustment we employed a simple polynomial functional form for the zero coupon yield curve. This method, not only provided a good fit to the data, but also produced a plausible implied zero coupon forward rate curve and a zero coupon volatility structure.

It was concluded that while the zero coupon method of adjusting coupon rates may be slightly more complex to implement, it was preferred because, it provided a slightly better estimate bond prices and yields, and as expected, it gave a very much better estimate of spot zero coupon rates that in turn could be used to provide zero coupon forward rate and volatility estimates.
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Introduction
There is a long and rich tradition of study of the term structure by both academics and practitioners. Their interest in the analysis of rates and yields is natural enough. In financial practice and decision making, there are no other variables that are as important or as pervasive as interest rates. A sound knowledge of the yield curve and its dynamics is the basis for the pricing and hedging of complex financial instruments.

Yield curve modelling is a process that estimates the relationship between the yield on an investment instrument and the term (or tenor) of that investment instrument. The inputs to this process are observed pairs of yields and terms, taken from traded instruments, and the output of the process is an estimated functional form relating yield to term. Naturally, it is essential that each observation on a specific security provides a yield that has an unambiguous relationship with the term of the security. A security will however, only have an unambiguous length if it makes a single payment at time, t, i.e. if it is a pure discount security. In Australian financial markets, while single payment instruments, in the form of discounted bills and notes, are common for very short tenors they are virtually non-existent for maturities of more than a year or so.

The market for coupon paying bonds determines Australian longer term interest rates. This poses a problem for the yield curve modeller. The problem manifests itself in the so called coupon effect. The coupon effect depicts the observed phenomenon that bonds that are identical in all respects except the size of the coupons have different yields. Yield curve modellers that use yields from coupon, rather than zero coupon, bonds, must employ a modelling technique that takes account of the coupon effect.

The ambiguity in the relationship between the yield on coupon bonds and their term may be rectified by either,

1. adjusting the term so that it more accurately reflects the effective length of the underlying bond or

2. by adjusting the yield so that it more accurately represents the yield of a pure financial instrument of the term in question.

This paper examines the above two alternative methods of fitting yield curve models to market determined yields on coupon paying bonds. The approach that underlies these alternate methods of adjusting the coupon yield, term data, is schematically illustrated in Figure 1.
The first of these methods (the duration method) uses yields on coupon paying bonds but adjusts the length of the bond to reflect the average time that a coupon bond makes its payments. In the duration based method, market yields on coupon paying bonds are modelled as functions of the duration of the bond, rather than as functions of the term of the bond.

The second method of modelling yields on coupon paying bonds (the zero coupon method) utilises a technique for deducing zero coupon yields from coupon yields. In this method the deduced zero coupon bond yields are modelled against the actual term of the bond.

Yield curve models have invariably always embraced the principle of parsimony. Good empirical models are (1), relatively simple, with a parsimoniously small number of parameters and yet (2), fit the observed data smoothly and well. Both the duration and zero coupon techniques, explored here, employ a four parameter model. By doing so, each embraces the principle of parsimony of structure.

This paper makes a number of original contributions. The most important of these is that it provides an explicit comparison of the two most common methods of dealing with the coupon effect. Any practitioner, using coupon bond data to make term structure inferences, must make adjustments to the raw data in order to account for the coupon effect. This paper by comparing two alternative methods provides an insight into the most efficacious approach to the problem.

In estimating a duration based yield curve model we employ a variable to account for bond convexity. In our opinion, any duration based yield curve model ought to include a variable to reflect the attractiveness of bond convexity in the eye of the bond holder. To our knowledge this is the first time a convexity measure has been tested in a duration based yield curve model.

In estimating a zero coupon model we employ a simple polynomial functional form for the zero coupon yield curve that not only produces a good fit to the data but also
provides an easy insight into the implied zero coupon forward rate curve and the zero coupon volatility structure. While Adams and Deventer [1994] have applied a similar model to swap rate data this is the first time such a model has been applied to bond data.

Finally, this study employs a hitherto unexploited detailed data set on Australian traded bonds. Consequently this study adds a significantly to the body of knowledge on historical behaviour of Australian interest rates.

Data
The data used in this study consists of daily observations on the yields on Australian Treasury bonds for the period January 1, 1988, to October 30, 1991. This data set resulted from a merging of two subsets of data; the first was taken from published yields in the Australian Financial Review. The second data set was provided by the Australian Gilt Securities Limited. The merged data set is unique and has not previously been used in any yield curve study. For the purposes of this study, the data was contracted to weekly observations by the extraction of Wednesday market data. This resulted in a total of 3,587 total observations on yield and tenor divided into 198 weekly yield curves. The 198 weekly term structures had on average about 18 observations.

The Figures 2 and 3 below afford some insight into the nature of the data. Figure 2 is a frequency histogram of the term of the observations. The term of the bonds in the study ranged from the shortest of length less than one year to the longest of length 14 years.

There was considerable movement in yields over the four years of the study. Figures 3 and 4 show that yield movement occurred in the general level of interest rates and in the relationship between short and long term rates.
Figure 3 shows that interest rates were relatively low at the beginning and at the end of the sample and that at these times the yield curve was normally shaped. During the middle of the sample period the interest rates were relatively high and the yield curve was inverse in shape. These observations are confirmed by Figure 4. Figure 4 plots the weekly term structure for the first week for the four years of the study.
Duration Based Yield Curve Model

MacCaulay's duration for a general multi-payment financial instrument, such as a coupon paying bond, is defined as the weighted average of the time of payment. A pure financial instrument makes a single payment at the termination of its life. The duration of a pure financial instrument is thus identical to its term. However, the duration for a coupon paying bond will necessarily be shorter than its term. It can be argued that the duration of a coupon paying bond, being a measure of the average time of payment, represents the effective term or tenor of an equivalent or underlying hypothetical pure security. Further it is argued that many of the properties of a coupon bond are a reflection of the properties of its zero coupon equivalent.

If one accepts the argument that the duration of a coupon paying bond is a measure of its effective term, it follows that one ought to model the yields on coupon paying bonds as a function, not of bond tenor, but rather of bond duration. This is the approach taken by Hsu and Chu[1993]. They model the yields on US Government securities as a function of the duration of the securities.

Hsu and Chu employed a, four parameter, non-linear Laguerre function. The use of this particular functional form was pioneered by Lau[1983] and Nelson and Siegel[1987]. While the flexibility of Laguerre functions makes them attractive forms for yield curve modelling, both Hall[1989] and Bhar and Hunt[1993] found that they have their drawbacks in empirical application. Both of these studies found that simple polynomial models outperformed the Laguerre function models in empirical yield curve applications.

The duration model, employed in this study, posits that the yield on a coupon paying bond is a polynomial function of its duration. The duration model is specified as,

\[ y(T_j) = \beta_1 + \beta_2 d_j + \beta_3 d_j^2 + \beta_4 c_j \]

\[ j = 1...m \]

(1)

\( y(T_j) \) is the yield on the jth coupon paying bond with a term equal to \( T_j \), \( d_j \) is the bond's duration and \( c_j \) is a term representing the jth bond's convexity. The parameters, \( \beta_1 \ldots \beta_4 \), determine the shape of the yield curve.

The convexity term, \( c_j \), is included in (1) because this term modifies the response of bond's price to changes in yields. The well known second order approximation relating the percentage change in a bond's price to a change in yield to maturity is,

\[ \frac{\Delta P}{P} \approx -\ddot{d} \Delta y + c \Delta y^2 \]

(2)

where \( P \) is the price of a bond and \( \ddot{d} \) is the bond's modified duration. As the convexity, \( c \), is invariably positive, a higher value of convexity makes a bond more attractive to a risk averse investor. A more convex bond outperforms a less convex bond whenever yields change. When yields fall, the price of a more convex bond rises more than the price of a less convex bond of similar duration. When yields rise, a more convex bond falls less than a less convex bond.

The attractiveness of convexity implies that \( \beta_4 \) in equation (1) ought to be negative. All other things being equal, an investor will be prepared to accept a lower yield on a more convex bond. Naturally we expect that, \( \beta \), that is an estimate of the short term interest
rate, ought to be greater than zero. It is expected that $\beta_1$, the yield curve slope coefficient, will fluctuate around zero. $\beta_2$ will be positive whenever the yield curve is normally shaped and will be negative whenever the yield curve is inverse. The parameter $\beta_3$ determines the amount of curvature in the estimated model.

Yield in equation (1) is a simple linear function of the explanatory variables and thus can be estimated by OLS regression. Indeed the simplicity of the duration based yield curve model is one of its most appealing features when compared to the alternative zero coupon model yield curve model. The results of estimation are summarised in Table 1.

<table>
<thead>
<tr>
<th>Table 1 : Results of Estimation of the Duration Yield Curve Model</th>
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<tr>
<td>Statistics</td>
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<tr>
<td>Minimum</td>
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<tr>
<td>Maximum</td>
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<tr>
<td>Average</td>
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<tr>
<td>Proportion Significant at 5% level</td>
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The fit of the duration based model to the weekly term structure data was generally good. The average $R^2$ over the 198 estimated yield curves was 87.4%. Figure 5 contains market bond yield data and the fitted models for two dates, 5th July 1989 and the 30th October 1991.

<table>
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<th>Figure 5 : Fitted Duration Based Yield Curve Models</th>
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It is worth noting the failure of the convexity term to provide any significant explanatory power. The estimated parameter associated with convexity, $\beta_4$, was very
often of the wrong sign. In any case it can be seen from Table 1. that $\beta_4$ was only significant in 2% of cases (4 of 198 estimated weekly equations).

There are two possible explanations for the lack of significance of convexity in the estimated models. The first is that the Australian investor, in considering holding period risk, simply does not care about the second order price effects embodied in the convexity statistic. The second explanation is that the investor does care about second order effects however there is not sufficient variety in convexity to allow the investor to make a meaningful choice between bonds on this basis. There is perhaps some merit in this second argument. If one considers a 1% p.a. change in yield then the change in the percentage price of the jth bond due to convexity per unit of duration is,

$$\frac{c_i}{100d_i}$$

Over the 3,587 observations the value of this variable averaged 5.05% p.a. with a standard deviation of 2.11%. That is, over the entire data set there was only a range of about an 8% difference in the bonds proportional price response per unit of duration.

In summary it can be said that the duration based yield curves fitted the data well but that convexity provided no significant input in determining the yields on coupon paying bonds.

**Zero Coupon Yield Curve Model**

During its life a coupon bond makes a number of cash payments, one for each of the coupons and finally at maturity a redemption payment. Thus a coupon paying bond can be thought of as being of a bundle of zero coupon bonds, one for each of the bond’s cash payments. This line of reasoning leads to the conclusion that the yield on a coupon paying bond is a weighted average of the yields on each of its constituent zero coupon bonds. The market yield on a coupon bond will be the weighted average of the each of yields on the coupon bond’s constituent zeros, the longest of which has a term of length t. Thus, as long as the yield curve is not flat, there will always be a difference between the t period zero coupon yield and the yield on a t period coupon bond.

It also follows that as long as the yield curve is not flat the yield on each of two bonds, that are identical in length but differ as to their coupon rates, must also differ. A high coupon bond will have a yield that is appropriate to a shorter term while a low coupon bond will have a yield that is more aligned to a longer term. This so called *coupon effect* invalidates the practice of constructing a yield curve by simply plotting the yields on coupon bonds against the term of the bonds.

Early yield curve researchers ignored the coupon problem and treated the yields on coupon bonds as if they were zero coupon yields (See Durand[1942] and Nelson[1972]). Later studies, that acknowledged the coupon problem, exploited various techniques whereby a bond’s coupon, were conceptually "stripped off" to reveal the yield on the underlying zero coupon bond. McCulloch[1975] and Coleman, Fisher and Ibbotson[1992], Vasicek and Fong [1982] and Adams and van Deventer [1994]. are examples of yield curve studies that have used coupon stripping techniques.
The process of fitting a model of the term structure to market data on coupon paying bonds may be summarised as follows. It is assumed that the price of a semi-annual coupon bond is the sum of its discounted cash flows. That is,

\[ P_j(T) = \sum^{2T}_{i=1} e^{-y(t/2) i/2} + e^{-y(T)T} F \quad j=1...m \]  

(4)

where, \( P_j(T) \) is the price of the jth bond with term, \( T \), \( y(t) \) is the continuously compounded, zero coupon, interest rate, appropriate to term, \( t \). \( C \) and \( F \) are the bond’s cash flows, representing the coupon payments and the face value respectively. The yield curve modeller’s task is to produce a “good” estimate of \( y(t) \). In undertaking this task the researcher must take a number of decisions.

First, the yield curve modeller must decide the criterion, against which, the goodness or otherwise of a particular yield curve model will be measured. The Vasicek and Fong[1977] and Adams and Deventer[1994] studies are recent examples in a tradition of yield curve researchers who have used the “smoothness” of the fitted yield curve splines as the criterion for selecting a particular yield curve model. We have not taken this approach. Our approach has been to fit estimated coupon bond prices, as implied by a zero coupon yield curve model to the market data using nonlinear least squares. That is, we use the sample data to select a form for \( y(t) \) that minimises the sum of squared differences between the predicted price of a coupon bond, and its market price.

The next decision a yield curve modeller must take is whether to model the spot yield curve, \( y(t) \), or the forward yield curve, \( f(t) \). The choice is arbitrary in that one can always derive the spot curve from the forward or vice versa as,

\[ y(t) = \frac{1}{t} \int_0^t f(s) ds \]  

(5)

\[ f(t) = y(t) + ty'(t) \]  

(6)

There is an argument for modelling the forward curve as one can directly apply restriction to the shape of the forward curve. For example, the restriction that the forward rate stabilises at the long end of the curve is seen as desirable imposition. From our perspective however the relative ease of estimation of the spot curve compared to the estimation of the forward curve, outweigh any other considerations.

The last decision a yield curve modeller must make concerns the specification of the yield curve function. A number of researchers have used various forms of exponential curves such as Laguerre functions. Our previous experience of modelling term structures convinced us that a four parameter polynomial function would provide the better results. Thus the particular form of the fitted yield curve yields was,

\[ y(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3 \]  

(7)

As our raw yield data consisted of market rates quoted on the basis of semi-annual compounding, we decided to estimate the zero coupon yield curve on the same basis. Estimation consisted of selecting values for the B parameter vector (\( \beta_1, ..., \beta_4 \)) such that we minimised,

\[ \sum_{j=1}^{n} (P_{\text{market}} - P(y(B, t))^2 \]
where,
\[
P_j(y(B,T)) = \frac{C}{\left(1 + y(B, T/2)/2\right)^i} + \frac{F}{\left(1 + y(B, T)/2\right)^{2T}}
\]

is the predicted value of the jth bond and \( P_{\text{market}} \) is the market price of the bond.

The yield curve was estimated for 198 weeks using the TSP econometric package. The results are summarised in Table 2 and Figures 6 and 7.

**Table 2 : Results of Estimation of the Zero Coupon Yield Curve Model**

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Parameter Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \beta_1 )</td>
</tr>
<tr>
<td>Minimum</td>
<td>8.4805</td>
</tr>
<tr>
<td>Maximum</td>
<td>16.5345</td>
</tr>
<tr>
<td>Average</td>
<td>12.5624</td>
</tr>
<tr>
<td>Proportion Significant at 5% level</td>
<td>100.00%</td>
</tr>
</tbody>
</table>

The zero coupon model fit was generally good as is indicated by the high R² numbers. There is however strong evidence that the model is over specified as \( \beta_3 \) is statistically insignificant for the vast majority of weeks. We have however retained the \( t^t \) term in our model in order that the zero coupon results be comparable with the results of the, four parameter, duration based model.

An example of the fitted zero coupon yield curve for the last week in the sample period is set out in Figure 6. Note that the zero coupon yield curve lies above the coupon yield as is expected with an upward sloping term structure.

**Figure 6 : Estimated Zero Coupon Yield Curve**

![Estimated Zero Coupon Yield Curve](image-url)
A three dimensional representation of the fitted zero coupon yield curve is contained in Figure 7.

One of the advantages of the zero coupon yield curve model, compared to the duration model, is that the zero coupon model can be easily transformed to provide a models for,

(1) the forward rate term structure and
(2) the spot rate volatility term structure.

A zero coupon forward rate, $f(t)$, is defined as the one period interest rate on a security that begins in life a time $t$. Assuming semi-annual compounds, the forward rate structure can be deduced from the spot rates as,

$$f(t) = 2 \left( \frac{(1 + r(2t+1)/2)^{2t+1}}{(1 + r(2t)/2)^{2t}} - 1 \right)$$

(9)

The implied forward rate structure over the sample period is presented in Figure 8. It can be observed from Figure 8 that the forward rate term structure is reasonably well behaved and does not strongly contradict a priori expectations about the stability of the forward rate structure.
The volatility of spot rates is easily estimated from the series of weekly zero coupon yield curves. The standard deviation of the yield of term \( t, \sigma_{y(t)} \), can be computed as:

\[
\sigma_{y(t)} = \tau(t)' \Omega \tau(t)
\]

where \( \tau(t)' \) is a vector \( (1, t, t^2, t^3) \) and \( \Omega \) is a 4x4 covariance matrix of the estimated parameters \( \beta_1, \beta_2, \beta_3, \beta_4 \). The term structure of zero coupon spot rate volatility is plotted in Figure 9. The model produces a volatility term structure that is consistent with the prior belief that long term rates are more stable than short term rates.
Comparison of the Duration and Zero Coupon Models

The obvious test of the goodness of a term structure model is to see how well it fits the observed bond market data. Bond market data may be expressed as either yields or price. Thus we tested the goodness of fit of the competing models in terms both price and yield. It is of course possible that one or other model provides a superior performance, not along the whole yield curve, but rather over sections of the yield curve. To check for this possibility we ascertained the fit of each model, as measured by root mean squared percentage error (RMSPE), at yearly intervals along the term structure.

Three models were tested for goodness of fit. In the first model (the zero model) bond prices were computed by discounting each of a bond’s cash flow using the predicted zero coupon interest rate appropriate to the term of the cash flow. Once a bond’s zero coupon model price had been computed by this method, its implied yield, based on that price, was calculated.

The second model (the duration model) takes as its basis yields, as predicted by the duration model. In this model the predicted yields are used to deduce duration model predicted bond prices via the yield to maturity pricing model. Here, each of a bond’s cash flows are discounted using the same, duration model predicted, yield.

The third model (the zero/duration model) is a combination of the first two models. The zero/duration model takes the predicted yields from the duration model to individually discount a bond’s cash flows as if they were zero coupon rates. If in fact the substitution of duration for term provides a workable alternative to direct zero coupon curve estimation (as illustrated in Figure 1) the fit of this third model ought to rival that of the first.
| Fitted Variable | Prices | | | Yields | | | |
|-----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Model           | (1) Zero | (2) Duration | (3) Zero / Duration | (1) Zero | (2) Duration | (3) Zero / Duration |
| Tenor           |               |               |               |               |               |               |
| 1               | 0.22%         | 0.17%         | 0.97%         | 1.20%         | 0.80%         | 4.39%         |
| 2               | 0.20%         | 0.19%         | 1.37%         | 0.77%         | 0.74%         | 5.03%         |
| 3               | 0.24%         | 0.20%         | 2.44%         | 0.70%         | 0.60%         | 6.48%         |
| 4               | 0.21%         | 0.23%         | 7.18%         | 0.53%         | 0.56%         | 13.24%        |
| 5               | 0.21%         | 0.21%         | 13.58%        | 0.44%         | 0.43%         | 27.44%        |
| 6               | 0.23%         | 0.26%         | 18.64%        | 0.42%         | 0.49%         | 37.19%        |
| 7               | 0.27%         | 0.28%         | 21.97%        | 0.46%         | 0.48%         | 46.95%        |
| 8               | 0.28%         | 0.35%         | 24.23%        | 0.44%         | 0.56%         | 51.45%        |
| 9               | 0.29%         | 0.42%         | 24.62%        | 0.45%         | 0.66%         | 50.91%        |
| 10              | 0.39%         | 0.43%         | 23.80%        | 0.55%         | 0.62%         | 46.33%        |
| 11              | 0.31%         | 0.42%         | 23.58%        | 0.42%         | 0.56%         | 43.18%        |
| 12              | 0.27%         | 0.36%         | 21.51%        | 0.36%         | 0.46%         | 40.84%        |
| 13              | 0.36%         | 0.32%         | 26.97%        | 0.46%         | 0.40%         | 42.72%        |

The RMSPE figures in Table 3 clearly demonstrate that the inferiority of the zero/duration model’s performance in predicting either bond yields or prices. The inadequacy of this models performance can be traced the inability of the duration model yields to substitute for zero coupon rates. This inadequacy is illustrated in Figure 10.

The performance of the duration and the zero coupon models are roughly comparable. The zero coupon model outperforms the duration model for 8 of the 13 observations on both price and yield. The difference however, was not statistically significant. The performance of each model varies over the length of the yield curve with the duration model’s performance is relatively better at the short end of the term structure.
Conclusion
At the heart of this paper is a comparison of the performance of two alternative methods adjusting data on coupon bonds to account for the coupon effect. Of these two methods, the duration based approach is superficially more attractive because of its ease of implementation. The results however, show that this method is slightly inferior in its ability to provide a fit to bond prices and yields when compared to the alternative zero coupon method. What is clear from the results is that the duration method fails dismally in providing good implied estimates of the underlying zero coupon interest rates.

It is concluded that while the zero coupon method of adjusting coupon rates may be slightly more complex to implement, it is preferred because, (1) it provides a slightly better estimates of bond prices and yields and (2) as as expected, it gives a very much better estimate of spot zero coupon rates that in turn can be used to provide zero coupon forward rate and volatility estimates.
References


Hall, A.D. "Australian Yield Curves", mimeograph, Australian National University


1 Laguerre functions consist of a polynomial multiplied by an exponential term. A typical Laguerre function such as that used by Nelson and Seigal[1987] is

\[
y(t) = \beta_1 t + \beta_2 t_2 + \cdots + \beta_m t_m \exp(-\beta_0 t) + \beta_4, \quad I = 1, \ldots, m
\]

where \( t_i \) is the term of the \( i \)th bond.

2 These dates were chosen as being representative of an inverse and a normal term structure respectively.

3 A detailed survey of yield curve modelling approaches is provided by Buono, Gregory-Allen and Yaari[1992].

4 It was assumed, for estimation purposes that all bonds had a face value of $100.