Fitting Parsimonious Yield Curve Models to Australian Coupon Bond Data

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by

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Abstract

This study uses a unique data set on Australian coupon bonds to test a number of yield curve models. A non-linear least squares technique is employed to directly fit four alternative, zero coupon, forward rate, yield curve models to the data. The four yield curve models tested were two, 4-parameter polynomial curves and two 3-parameter models including a Laguerre function. We show that a fourth order polynomial with the cubic term omitted best fits the data. This preferred model provides good estimates of both the forward and spot rate curves as well producing volatility structures that accorded with our a priori expectation.

The preferred, fourth order polynomial model is used as the basis of a market trading strategy. In this strategy, model predicted underpriced bonds are purchased and a model predicted overpriced bonds are sold. This strategy is shown to be superior to a random trading strategy. There is however, little evidence of market inefficiency as transaction costs account for any profit generated by the strategy.
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Introduction

Yield curve modelling is a process that estimates the relationship between the yield on an investment instrument and the term (or tenor) of that investment instrument. The inputs to this process are observed pairs of yields and terms, taken from traded instruments, and the output of the process is an estimated functional form relating yield to term. Naturally, it is essential that each observation on a specific security provides a yield that has an unambiguous relationship with the term of the security. A security will however, only have an unambiguous length if it makes a single payment at time, t, i.e. if it is a pure discount security. In Australian financial markets, while single payment instruments, in the form of discounted bills and notes, are common for short tenors they are virtually non-existent for maturities of more than a year.

The market for coupon paying bonds determines Australian longer term interest rates. This poses a problem for the yield curve modeler. The problem manifests itself in the so called coupon effect. Bonds that are identical in all respects except the size of the coupons will have different yields.

The ambiguity in the relationship between the yield on coupon bonds and their term may be rectified by estimating the underlying zero coupon yields from the coupon bond yields. Text books routinely illustrate the so called bootstrap method of estimating a zero coupon yield curve from a coupon bond yield curve. In this method coupons are sequentially “stripped” from a bond, leaving only the final payment, and thus an implied zero coupon rate.

The bootstrap method however unrealistically simple. It assumes that a yield curve modeler has access to bond data where bond terms are uniformly spaced one coupon period apart. In practice bond terms are not uniformly spaced and there are frequent examples of inconsistent, or contradictory, yields in bond market data. Inconsistent yields arises whenever virtually identical bonds trade at different yields. In this paper we demonstrate the use of maximum likelihood technique that copes with market data on non-uniformly spaced bonds, with possible contradictory market yields, that simultaneously strips the coupons and estimates zero coupon spot and forward rate yield curves.

The paper makes a number of original contributions. First, it sets out a method of direct estimation of the zero coupon forward rate curve using coupon bond data. Secondly it applies this method to a unique data set of market yields and terms on Australian coupon bonds. Thirdly it compares the performance of various functional forms in modelling yield curves. Finally the estimated yield curve models are used in test of the efficiency of pricing in the market for Australian coupon bonds.

1 See Fabozzi and Fabozzi(1989) or Hunt and Terry (1992) for worked examples of the bootstrap method.
Data

The data used in this study consists of daily observations on the yields on Australian Treasury bonds for the period January 1, 1989, to November 12, 1994. This data was kindly provided by Australian Gilts Securities. The data set is unique and has not previously been used in any yield curve study. For the purposes of this study, the daily data was reduced to monthly observations by the extraction of market data for each last Wednesday of each of the months of the study. Wednesday was chosen as the monthly focal date because, of all days of the week, it is least affected by public holidays. The last Wednesday was used as this date avoids the potential problem of a change in the ex-interest convention that occurred during the period of the study. The extraction process resulted in the production of 70 monthly yield curves containing a total of 1,723 observation pairs of yield and tenor. The 70 monthly term yield curves consisted on average of about 25 observations on yield and term. Summary statistics on bond terms, coupon rates and number of observations per month are set out in Table 1.

<table>
<thead>
<tr>
<th>Table 1: Data Summary Statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term (Years)</td>
</tr>
<tr>
<td>Minimum</td>
</tr>
<tr>
<td>Average</td>
</tr>
<tr>
<td>Maximum</td>
</tr>
</tbody>
</table>

There was considerable movement in yields over the five years of the study. Figures 1 and 2 show that yield movement occurred in the general level of interest rates and in the relationship between short and long term rates.

Figure 1: Interest Rate Dynamics

![Graph showing interest rate dynamics](image)

The overall level of interest rates is depicted in Figure 1 by the yield on a short bond. The slope of the yield curve is illustrated in the same figure as the premium of 10-year...
yields over 1-year yields. Figure 1 shows that interest rates were relatively high during
the first two years of the sample. During this time the term structure of interest rates
was inverse in shape. For the remaining three years of the study the yield curve was
normal in shape. Interest rates while falling for the most part of the study, rose
considerably in the last year of the study. These observations are confirmed in Figure 4
where the term structures for the first month of each year of the sample period are
plotted.

**Figure 2:**

<table>
<thead>
<tr>
<th>Year</th>
<th>Tenor (Years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>5.0</td>
</tr>
<tr>
<td>10.0</td>
<td>15.0</td>
</tr>
</tbody>
</table>

- 25.01.89
- 31.01.90
- 30.10.91
- 29.01.92
- 27.01.93
- 19.01.94

**Zero Coupon Yield Curve Model**

During its life a coupon bond makes a number of cash payments, one for each of the
coupons and finally at maturity a redemption payment. Thus a coupon paying bond
can be thought of as being of a bundle of zero coupon bonds, one for each of the bond’s
cash payments. This line of reasoning leads to the conclusion that the yield on a coupon
paying bond is a weighted average of the yields on each of its constituent zero coupon
bonds. The market yield on a coupon bond will be the weighted average of the each of
the yields on the coupon bond’s constituent zeros, the longest of which has a term of
length t. Thus, as long the yield curve is not flat, there will always be a difference
between the t period zero coupon yield and the yield on a period coupon bond.

It also follows that the yield on each of two bonds, that are identical in length but differ
as to their coupon rates, will also differ whenever the yield curve is not flat. A high
coupon bond will have a yield that is appropriate to a shorter term while a low coupon
bond will have yield that is more aligned to a longer term. This so called coupon effect
invalidates the practice of constructing a yield curve by simply plotting the yields on
coupon bonds against the term of the bonds.

Early yield curve researchers ignored the coupon problem and treated the yields on
coupon bonds as if they were zero coupon yields (See Durand[1942], Nelson[1972]) and
Frishling et al (1994). Later studies, that acknowledged the coupon problem, exploited various techniques whereby a bond’s coupons were conceptually “stripped off” to reveal the yield on the underlying zero coupon bond. McCulloch[1975] and Coleman, Fisher and Ibbotson[1992] are examples of yield curve studies that have used coupon stripping techniques. The process of fitting a model of the term structure to market data on coupon paying bonds may be summarised as follows.

It is assumed that the price of a semi-annual coupon bond is the sum of its discounted cash flows. That is,

\[ P_j(T) = C \sum_{t=t}^{T} e^{-y(t) t} + F e^{-y(T) T} \quad j=1...m \]

where, \( P_j(T) \) is the price of the jth bond with term, T, paying its next coupon at time, \( t \), \( y(t) \) is the continuously compounded, zero coupon, interest rate, appropriate to term, \( t \). \( C \) and \( F \) are the bond’s cash flows, representing the coupon payments and the face value respectively. The yield curve modeller’s task is to produce a “good” estimate of \( y(t) \). In undertaking this task the researcher must take a number of decisions.

First, the yield curve modeller must decide the criterion, against which, the goodness or otherwise of a particular yield curve model will be measured. The Vasicek and Fong[1977] and Adams and Deventer[1994] studies are recent examples in a tradition of yield curve researchers who have used the “smoothness” of the fitted yield curve splines as the criterion for selecting a particular yield curve model. We have not taken this approach. Our approach has been to fit estimated coupon bond prices, as implied by a zero coupon yield curve model to the market data using nonlinear least squares. That is, we use the sample data to select a form for \( y(t) \) that minimises the sum of squared differences between the predicted price of a coupon bond, and its market price.

The next decision a yield curve modeller must take is whether to model the spot yield curve, \( y(t) \), or the forward yield curve, \( f(t) \). The choice is arbitrary in that one can always derive the spot curve from the forward or vise versa as,

\[ y(t) = \int_{t_0}^{t} f(s) ds \quad \text{and} \]

\[ f(t) = y(t) + ty'(t) \]

In this study we have chosen to estimate the forward rate curve. This allows us to directly specify the shape of the forward rate and in some instances to apply desirable restrictions to that shape. Many dynamic term structure models are specified in terms of the forward rate (see for example Heath Jarrow and Morton (1992). Direct estimation of the forward rate curve facilitates a comparison of empirical results with the a priori propositions of forward interest rate structure theory.

The last decision a yield curve modeller must make concerns the specification of the yield curve function.

The first attempts to model yield curves used simple graphical interpolative techniques, e.g., Durand[1942]. McCullough [1975] advanced this approach through his use of a polynomial spline technique. Many researchers have employed this technique, for example, Vasicek and Fong [1982] and Adams and van Deventer [1994]. Spline curve fitting produces a continuously smooth curve that passes through every important data point. Spline curve fitting necessarily involves the estimation of many parameters. In
contrast to this multiple parameter approach, many yield curve models have frequently embraced the principle of parsimony.

Under the parsimony principle, empirical models are (1), relatively simple, with a parsimoniously small number of parameters and yet (2), fit the observed data smoothly and well. In this study we employ a series of three and four parameter model. Each of these simple low parameter models clearly embraces the principle of parsimony of structure.

there is an alternative tradition of using non linear least square's techniques to select the best yield curve model. Here yield curve models are estimated by regressing either bond yields or prices against a small number of explanatory variables. See for example, Nelson and Siegal [1987], Hsu and Chu [1993], Hall [1989] and Bhar and Hunt [1993]. A detailed survey of yield curve modelling approaches is provided by Buono, Gregory-Allen and Yaari[1992].

In this study we compared the performance of four separate forward rate models. These were

\begin{align*}
\text{Model 1} & \quad f(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3 \\
\text{Model 2} & \quad f(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^4 \\
\text{Model 3} & \quad f(t) = \beta_1 + \beta_2 e^{\beta_3 t} \\
\text{Model 4} & \quad f(t) = \beta_1 + \beta_2 t + \beta_3 / (t + 1)
\end{align*}

Where \( f(t) \) is the continuously compounded forward rate, \( \beta_i - \beta_i \) is a vector of yield curve level and shape parameters and \( t \) is the term in years.

Models 1 and 2 are four parameter polynomials of term. Polynomial yield curve models have a number of practical advantages over alternative models. Polynomial models have the same basic structure for both the forward and the spot yield curves.(see equations (8) and (9)). Polynomial models are linear in the variables. These two properties ensure that the polynomial form is relatively easy to estimate compared to alternative non-linear specifications. Moreover, complex term structures such as "humps" or "trenches" are easily accommodated within a four parameter polynomial structure.

Our previous experience of modelling term structures convinced us that a four parameter polynomial function, such as Model 1, would provide good empirical results.\(^2\) The use of a fourth order polynomial with the cubic term missing (Model 2) was inspired by the success achieved by Adams and van Deventer(1994) using this to model swap forward rates.

A number of researchers have previously used Laguerre functions to model yield curves.\(^3\) This study found, as others had, that a four parameter Laguerre function was empirically intractable. Multicolinearity between the regressor variables in the four

\(^2\) See Bhar and Hunt (1994)

\(^3\) For example see Nelson and Siegal (1987). Laguerre functions are of the general form, \( y = \alpha_0 + (\sum_{i=0}^{n-1} \alpha_i x^i) e^{x} \)
parameter Laguerre model regularly prevented convergence of the maximum likelihood estimator. Reluctantly we settled on the use of a three parameter Laguerre function, Model 3, to model yield curves. In the Laguerre form the forward rate asymptotes to \( \beta_i \) as the term increases. This restriction accords with both interest rate theory and empirical observation.

Statistical theory dictates that the more parameters a model has the better will be its fit to a data set. Hence we would expect that a four parameter model, like Models 1 or 2, would outperform a three parameter model like Model 3. Accordingly, we chose another three parameter model, Model 4, against which the performance of the Laguerre model could be measured.\(^4\)

Estimation consisted of three steps. First, for a particular model, zero coupon forward rates were converted to zero coupon spot rates. Next, these spot rates were used to compute a model implied price for each of the bonds in the yield curve. Last, the sum of squared differences between the model implied prices and the actual market prices (as implied by market yields) was calculated. This sum of squares was minimised using the TSP econometrics package to produce the estimated yield curve.

The spot zero coupon yield curves were derived from their forward rate counterparts ((4)-(7)), using (2), as

\[
y(t) = \beta_1 + \frac{\beta_2 t}{2} + \frac{\beta_3 t^2}{3} + \frac{\beta_4 t^3}{4}
\]

Model 1

\[
y(t) = \beta_1 + \frac{\beta_2 t}{2} + \frac{\beta_3 t^2}{3} + \frac{\beta_4 t^4}{5}
\]

Model 2

\[
y(t) = \beta_1 + \left( \frac{\beta_2}{\beta_3} e^{\beta_3 t} \right) / t
\]

Model 3

\[
y(t) = \beta_1 + \beta_2 t + \beta_3 \ln(t + 1)
\]

Model 4

Estimation consisted of selecting values for the \( \beta \) parameter vector (\( \beta_1, \ldots, \beta_{n} \)) such that we minimise,

\[
\sum_{j=1}^{m} \left( P_{j}^{\text{mkt}} - \hat{P}_{j}(f(B)) \right)^2
\]

where,

\[
\hat{P}_{j}(f(B)) = C \sum_{t=\tau}^{T} e^{-y(B,t)t} + e^{-y(B,T)}T \quad \text{and}
\]

\[
y(B,t) = \int_{0}^{t} f(B,s) \, ds
\]

with \( P_{j}^{\text{mkt}} \) being the market price of the \( j \)th bond with term \( T \).

\(^4\) Equation (7) is a three parameter version of Hall's (1989) functional form with the log term missing.
Estimation Results

The results of estimation are summarised in Tables 2, 3 and 4 and Figures 3, 4 and 5.

<table>
<thead>
<tr>
<th>Table 2: R² Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>R² Levels</td>
</tr>
<tr>
<td>Average</td>
</tr>
<tr>
<td>Standard Deviation.</td>
</tr>
<tr>
<td>Proportion of Best R²</td>
</tr>
<tr>
<td>Overall</td>
</tr>
<tr>
<td>Inverse Yield Curve</td>
</tr>
<tr>
<td>Normal Yield Curve</td>
</tr>
<tr>
<td>3 Parameter Class</td>
</tr>
<tr>
<td>4 Parameter Class</td>
</tr>
</tbody>
</table>

All of the models provided good fits with average R² figures above 95%. While Model 2 has the highest average R² figure and the lowest standard deviation its dominance is only marginal. It is the frequency with which it has the lowest overall R² for particular yield that sets model 2 apart from the other models. Model 2 provides the best overall fit for 40 of the 70 monthly yield curves.

The sample was separated into inverse and normal yield curves in order to see if the performance of a particular model was dependent on the shape of the yield curve. The results indicate that this is not so.

It was expected that the four parameter models would outperform the two three parameter models (Models 3 and 4). What was not expected was that the Laguerre model, Model 3, would do so poorly when compared with Model 4. The Laguerre model has the desirable property that longer run rates asymptote to a single figure. This aside, the results of estimation showed that there was no empirical reason for preferring the Laguerre model over the alternatives.

<table>
<thead>
<tr>
<th>Table 3: Proportion of Significant Model Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
</tr>
<tr>
<td>β₀</td>
</tr>
<tr>
<td>β₁</td>
</tr>
<tr>
<td>β₂</td>
</tr>
<tr>
<td>β₃</td>
</tr>
</tbody>
</table>

Table 3 shows that each of the estimated models contained a proportion of insignificant parameters. However, our approach was not a purely statistical one. We aimed to produce closely fitting yield curves using a small number of parameters. The contribution of a particular parameter to the fit of a particular yield curve depends on the shape of that yield curve. Sometimes the contribution will be significant at other times it will not be significant. In terms of our aims, there was no reason to exclude a parameter simply because it made a statistically insignificant contribution to the shape of many yield curves.

It is of course possible that one or other model provides a superior performance, not along the whole yield curve, but rather over sections of the yield curve. To check for this possibility we ascertained the fit of each model, as measured by root mean squared
percentage error (RMSPE) of the model predicted values, at yearly intervals along the term structure. The results are set out in Table 4.

<table>
<thead>
<tr>
<th>Term (Years)</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.53%</td>
<td>0.53%</td>
<td>0.62%</td>
<td>0.54%</td>
</tr>
<tr>
<td>2</td>
<td>0.59%</td>
<td>0.58%</td>
<td>0.74%</td>
<td>0.53%</td>
</tr>
<tr>
<td>3</td>
<td>0.58%</td>
<td>0.59%</td>
<td>0.79%</td>
<td>0.49%</td>
</tr>
<tr>
<td>4</td>
<td>0.58%</td>
<td>0.59%</td>
<td>1.03%</td>
<td>0.54%</td>
</tr>
<tr>
<td>5</td>
<td>0.56%</td>
<td>0.56%</td>
<td>1.34%</td>
<td>0.53%</td>
</tr>
<tr>
<td>6</td>
<td>0.59%</td>
<td>0.59%</td>
<td>1.45%</td>
<td>0.56%</td>
</tr>
<tr>
<td>7</td>
<td>0.54%</td>
<td>0.53%</td>
<td>1.70%</td>
<td>0.51%</td>
</tr>
<tr>
<td>8</td>
<td>0.47%</td>
<td>0.46%</td>
<td>2.08%</td>
<td>0.45%</td>
</tr>
<tr>
<td>9</td>
<td>0.45%</td>
<td>0.44%</td>
<td>2.42%</td>
<td>0.41%</td>
</tr>
<tr>
<td>10</td>
<td>0.59%</td>
<td>0.59%</td>
<td>2.73%</td>
<td>0.47%</td>
</tr>
</tbody>
</table>

Table 4 shows that with the exception of the Laguerre model the performance of the models are comparable. Further there does not appear to be any systematic relationship between a particular model and any section of the yield curve.

Preferred Model

Model 2, the 4th order polynomial with the cubic term missing, was selected as the preferred model to be used in further analysis. The shape of the estimated yield curve for this model at the beginning of each year of the sample period is set out in Figures 3 and 4.
It can be seen from Figures 3 and 4 that concerns about the instability of the fitted curves at the long end of the yield curve are without foundation.

A three dimensional representations of the fitted zero coupon forward rate yield curves are contained in Figure 5.
Volatility Structure

One can easily derive the volatility structure from the estimated forward rate yield curves. Figure 6 plots the estimated volatility term structure for the 1st month of each year from 1990 to 1994. Volatility, \( \sigma \), was estimated as the standard deviation of predicted forward rate for term, \( t \), over the previous 12 months. The various volatility structures appear to be consistent with the term structure dynamics proposed by Heath, Jarrow and Morton.\(^5\)

\[\text{Figure 6 : Estimated Forward Rate Volatility Structure}\]

A three dimensional representation of the volatility structure is presented in Figure 7.

\(^5\) Bhar and Chiarella (1994) show that the Heath, Jarrow and Morton term structure dynamics imply a Laguerre type volatility term structure.
A Trading Strategy

A particular zero coupon yield curve implies a unique price for every coupon bond. Thus if the bond market is (1) efficient and (2) the zero coupon yield curve is known, it follows that the price of every bond ought to equal its zero coupon predicted price. On the other hand, if the market is inefficient there may be persistent periods where bonds are over or under priced relative to their zero coupon predicted price.

We designed a simple trading strategy, based on the estimated zero coupon yield curve, in order to see whether a profit could be generated by the purchase of model predicted under priced bonds and the sale of model predicted overpriced bonds. The profit performance of this strategy is of interest as it comprises a joint test of the efficiency of the Australian bond market and the efficacy of the estimated polynomial zero coupon yield model.

To be more specific, our trading strategy involved a number of simple steps. First, for a particular month, the model predicted prices were computed. Next the single most overpriced and the single most under priced bonds were identified from the set of all bonds with tenor greater than three years and less than 12 years. Lastly, we computed a theoretical trading profit that would have arisen had we bought the under priced bond (the long bond) and sold the overpriced bond (the short bond) and subsequently closed

*Least squares estimation tends to produce its greatest residuals at the beginning and end of the estimation period. We excluded bonds at the beginning and the end of the term structure to prevent a contamination of our trading results by these residuals.
out these positions after one month. We normalised our trading profit by assuming that each long bond purchased had a face value of $1,000. The amount of the short bond traded was adjusted for any differences in the sensitivity of the two bonds to movements in interest rates. We assumed that the short bond sold had a face value of $1,000 multiplied by the modified duration of the long bond divided by the modified duration of the short bond.

The 70 month sample allowed us to make 69 buy/sell - sell/buy trades. A running proportion of winning trades is plotted in Figure 7. For much of the sample period the proportion of winning trades lies outside the two standard deviation limit.

The plot of the cumulative trading volume, Figure 8, shows that, ignoring transaction costs, the trading strategy was profitable. However, transaction costs as little as $0.18 per trade would account for all of the cumulative profit. Bearing this in mind, it is difficult to conclude that there existed significant unexploited trading opportunities. Of course a proponent of bond market inefficiency would however, point out that the trading time of one month is much too long to capture fleeting inefficiencies.
Summary

This study used a unique data set on Australian coupon bonds to test a number of yield curve models. A non-linear least squares technique was employed to directly fit four alternative, zero coupon, forward rate, yield curve models to the data. The results showed that of the estimated models, a fourth order polynomial with the cubic term omitted was best overall. This model provided good estimates of both the forward and spot rate curves as well producing volatility structures that accorded with our \textit{a priori} expectation.

The preferred, fourth order polynomial model was used as the basis for a test of a trading strategy and hence of bond market efficiency. In this strategy, model predicted underpriced bonds were purchased and a model predicted overpriced bonds were sold. While this strategy was shown to be superior to a random trading strategy. There was however, little evidence of market inefficiency as minimal transaction costs accounted for any profit generated by the strategy.
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Hall, A.D. “Australian Yield Curves”, mimeograph, Australian National University


