Keynesian Monetary Growth Dynamics: The Missing Prototype

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1 Introduction

In this paper we construct and analyse a nonstochastic descriptive Keynesian monetary growth model. The model consists of households (who are both workers and asset holders), firms (acting as production-units and investors) and government (fulfilling the role of fiscal and monetary authority). These economic agents interact on markets for goods, labour and financial assets, and we pay particular attention to model consistently the budget constraints linking these various economic agents and markets. Our model is a conventional Keynesian one of the standard textbook variety, though at a level of generality and consistency (with respect to budget constraints, investment formation and wage–price interaction) that has not previously been presented. Therefore, features which may seem indispensible from a more developed Keynesian perspective – for example approaches which give asset markets a much more dominant role than here – have not been specifically included. However our model establishes a consistent framework for such an integration. This means that a considerable amount of typical modern Keynesian questions must here be left for future research. We are convinced that a systematic treatment of Keynesian models of monetary growth which starts from orthodox presentations is of value in itself and will also help in the understanding of models existing in the literature which at first sight seem to be quite different in their structure.

Section 2 will present the general but nevertheless basic – Keynesian prototype model and explain its structure. We also derive in this section the reduced dynamic form of the model and determine its unique interior steady state solution. The simplified version of the model – where we neglect the GBC – will close this section. In section 3 we provide some basic considerations of the temporary equilibrium part of the model in general as well as of an important special case of it (where there is a given rate of interest in particular). This special case is employed in section 4 from the perspective of (and as part of) the real growth / real wage dynamics of the Goodwin model, here however with Keynesian demand problems. We shall find that the Goodwin type dynamics familiar from existing literature (see Chiarella and Flaschel (1994)) is considerably changed by the introduction of such demand problems. Section 5 then integrates the Rose (1969) type influence of the market for goods on the formation of the real wage and investigates from a local as well as from a global point of view the role of the stabilizing potential of flexible money wages in this general Keynesian prototype. Section 6 then integrates monetary phenomena into the real growth cycle models of sections 4, 5 by way of a money market dependent interest rate flexibility and its impact on investment behavior but keeping inflationary expectations fixed at their steady state value. We demonstrate here a variety of local (in)stability results including the loss of asymptotic stability at certain parameter values of the model via the occurrence of Hopf bifurcations which are already known from descriptive models of the Tobin and Keynes–Wicksell variety (see Chiarella and Flaschel (1994) and the other references cited there). Section 7 then expands the dynamic dimension of the model to four by allowing for adjustments in the expectations of medium run inflation and it considers the destabilizing potential of the Mundell effect in particular. Such a higher dimensional analysis has to remain very preliminary here, though its integration of the cycle generating mechanisms of the lower dimensional subcases appears
to be very interesting (see chapter 6 of Chiarella and Flaschel (1994) for some extensive numerical simulations on this most general step of our investigations.). Finally, section 8 offers some concluding comments.
2 A General Keynesian Prototype Model of Monetary Growth

We divide this section into three parts. In section 2.1 we lay out the economic definitions and relationships which form our general Keynesian model. In section 2.2 we derive the reduced form dynamical system to which the economic definitions and relationships of section 2.1 give rise. In this section we also discuss the equilibrium point of the reduced form dynamical system. In section 2.3 we point the inherent nonlinear structure of the model which occurs before the introduction of any nonlinear behavioural relationships.

2.1 The Economic Structure

As stated in the introduction our model is a conventional Keynesian one so that output of firms is determined by effective demand as in IS–LM models of textbook (or more elaborated) type. This means that we now have to distinguish between this output $Y$ and potential output (as in eq. (1) below). The resulting rate of capacity utilization will exercise influence on investment behavior as well as on price formation. These are the essential changes which distinguish our Keynesian model from models of the Keynes–Wicksell type. They may be small when looked at the extent of these changes, in particular when compared with the ones necessary to go from Tobin models to Keynes-Wicksell models. But they definitely improve the consistency of the Keynes-Wicksell approach significantly. One central question here is how much will in fact be changed in the quantitative responses of the system over time through the adoption of the indirect mechanism with respect to the consequences of goods-market disequilibrium (via the rate $U$) now on inflation as well as on investment in the place of the direct price mechanism of the Keynes-Wicksell model based on direct goods market disequilibrium.

Note that we assume fixed proportions in production to facilitate the analysis. The case of smooth factor substitution is treated in Chiarella and Flashel (1994) where it is demonstrated that such an addition does not modify our Keynesian prototype model in any essential way.

As in the case of the labor market and the there widely accepted concept of a natural (or non-accelerating rate of inflation) rate of employment $V < 1$, we now also assume as given a natural rate of capacity utilization $\bar{U} < 1$ which describes the benchmark between expansionary and contractionary investment and price effects. We assume furthermore that the distance of the maximum rate of capacity utilization $U_{\text{max}} = 1$ from the natural rate of capacity utilization $\bar{U}$ as well as the distance of the actual rate of employment $V$ from the maximum rate of employment (of the employed) $V_{\text{max}} > 1$ will both always stay positive so that the maximum rates can be neglected as limits in all following investigations. Of course, supply bottlenecks – as they can be generated by absolute limits on the employment of the labor force $L$ and by the capital supply $K$ – must be added eventually for a really complete
analysis of the model – by considering modifications of the assumed economic behavior when
the economy comes close to or is at $V_{\text{max}}$, $U_{\text{max}}$. Neokeynesian regime switching approaches
identify $\bar{U}$ with $U_{\text{max}}$ and $\bar{V}$ with $V_{\text{max}} = 1$ and thus claim that the steady-state path of a
capitalistic economy is always right at the border to such absolute supply bottlenecks.\footnote{We here instead use non-repressed inflation and buffers provided by the behavior of firms in order to
avoid switches away from the Keynesian regime to so-called Classical regimes or regimes of repressed inflation
- at least for a normal functioning of the economy around its steady state path.} Such
assumptions, however, do not represent a good description of the workings of a capitalist
economy.\footnote{Inventories provide a further buffer that is created by capitalist firms in order to circumvent the capital
stock constraint. This issue is taken up in Chiarella and Flaschel (1995b).}

1. Definitions (remunerations and wealth):

The equations (1),(2) of the model provide the definitions of important macroeconomic
magnitudes, namely of the real wage $\omega$, the wage share $u$ ($x$ gross output per laborer), the
actual rate of profit $\rho$, money $M$, bonds $B$, equities $E$ and real wealth $W$ which here consists
of real money balances, bonds and equity. Thus if we define $w$ as nominal wages, $p$ as
the price level, $Y$ as output, $K$ as capital stock, $L^d$ as employment, $\delta$ as depreciation plus
inventory rate, $\zeta$ as the real rate of interest (before taxes), $r$ as the nominal rate of interest
and $\pi$ as the expected rate of inflation, then\footnote{The variables used to describe the model are summarised in the appendix. Note in particular that lower
case letters are used to denote real variables in intensive form i.e. $y = Y/K$ etc. and nominal variables in
intensive form i.e. $m = M/(pK)$ etc.}

$$\omega = w/p, \quad u = \omega/x, \quad \rho = (Y - \delta K - \omega L^d)/K, \quad \zeta = r - \pi,$$  

(1)

and

$$W = (M + B + p_E E)/p, \quad p_B = 1.$$  

(2)

where $p_E, p_B$ denote respectively the price of equities and bonds.

2. Households (workers and asset-holders):

We assume two groups of households: workers and asset holders (capitalists) for whom we
are assuming Classical saving habits, i.e., workers do not save (constant savings propensity
$s_w = 0$) while capitalists save a constant fraction (constant savings propensity $s_c$) of their
perceived disposable income $Y^{D}_c$. Since we assume fixed proportions in production (in order
to simplify the analysis) we need such an assumption on differential savings habits in order
to get a steady state solution for such a model, since one generally does not exist for fixed
proportions technologies in the often assumed case $s_w = s_c = s$. This latter assumption
furthermore is less close to reality than the one we employ. Of course, assuming differential saving habits à la Kaldor would be even better (extended possibly also to other groups of savers such as pension funds for example). This extension is, however, left for future investigations.

Since wealth owners hold equities the real wealth component in asset owners’ portfolios is given by \( pE / p \) with respect to actual holdings as well as with respect to stock demand. Furthermore, the savings decision of capitalist households includes besides money and bonds the term \( pE^d / p \), i.e., that part of private savings that is intended to go into equities. The decision of asset holders concerns the allocation of their wealth between real balances (demand: \( M^d \)), government bonds (demand: \( B^d \)) and equities issued by firms (demand: \( E^d \)), as expressed in equation (3). Real capital is under the command of firms with regard to its use for production as well as with regard to its intended rate of change over time. So

\[
W = (M^d + B^d + pE^d)/p.
\]  

Note here that we assume that all taxes are paid by capitalists. Note furthermore that we do not distinguish between the actual and the perceived disposable income of capitalist households, so that we just use the simple income concept \( Y - \delta K + rB/p - T \) as perceived disposable income for the private sector as a whole. This income concept is based on production plans and not on actual sales. Of course, the discussion of more elaborate concepts of perceived disposable income could form the basis of future investigations of the model proposed here.

Equation (4) describes aggregate planned consumption \( C \) as the sum of the two components that we have already described above by referring to \( \omega L^d \) and \( Y^D_{c,s} \) as the perceived disposable income of workers and wealth owners respectively. Thus,

\[
C = \omega L^d + (1 - s_c)[\rho K + rB/p - T], \quad s_w = 0.
\]  

Letting \( T \) denote real taxes and \( Y^D_c \) actual disposable income of capitalists then total private savings, \( S_p \), is defined in equation (5) by total disposable income of households minus consumption i.e. \( \omega L^d + Y^D_c - C = Y - \delta K + rB/p - T - C \) and, of course, here is equal to the savings of capitalists out of their perceived disposable income \( Y^D_{c,s} \) (assumed equal to \( Y^D_c \)). Their intended allocation is determined by the flow magnitudes \( \dot{M}^d/p, \dot{B}^d \) and \( pE^d/p \). Thus,

\[
S_p = \omega L^d + Y^D_c - C = Y - \delta K + rB/p - T - C,
= s_c[\rho K + rB/p - T] = s_c Y^D_c,
= (\dot{M}^d + \dot{B}^d + pE^d)/p.
\]
It is assumed in equation (6) that normal labor supply \( L \) grows at a given rate \( n \) which is also equal to the growth rate of the total population. It is possible to allow for overtime work which means that the number \( L \) should not be interpreted as the maximum supply of labor hours available in the economy in the discussion that follows. Thus

\[
\hat{L} = n = \text{const.} \tag{6}
\]

3. **Firms (production-units and investors):**

Equation (7) describes the behavior of firms in the most elementary way possible in a two factor world. Firms produce output with a fixed proportions (linear) technology using capital and labor inputs. Using \( y \) \((y^p)\) to denote the actual (potential) output-capital ratio, \( YP \) potential output and \( U \) the rate of capacity utilisation we can write

\[
YP = y^p K, \quad y^p = \text{const.}, \quad U = Y/YP = y/y^p, \quad (y = Y/K) \tag{7}
\]

\[
L^d = Y/x, \quad x = \text{const.}, \quad V = L^d/L = Y/(xL). \tag{8}
\]

In this latter equation we use \( V \) to denote the rate of employment. In subsequent discussion we use \( \bar{V} \) to denote the employment complement of the non-accelerating inflation rate of unemployment - NAIRU.

We note that our model allows for varying degrees of utilisation \( U \) of the productive capacity of firms which feeds into investment behaviour. The investment function equation (8) assumes that investment per unit of capital is determined in a natural, possibly, nonlinear way by the differential between the net rate of return \( \rho^n \) on capital and the net rate of return \( \zeta^n \) on government bonds via the function \( i_1(\cdot) \) and the difference between capacity utilisation \( U \) and its NAIRU level \( \bar{U} \). We write this dependence as a linear function here though it could be made nonlinear in subsequent extensions. There is further a trend growth rate in this investment function which for simplicity is here set equal to \( n \) the labour supply growth rate. We thus write,

\[
I = i((1 - \tau)\rho - ((1 - \tau)r - \pi))K + i_2(U - \bar{U})K + nK. \tag{9}
\]

Firms issue equities in order to finance investment and they have by assumption no retained earnings with respect to their planned production and planned proceeds which means that firms have to issue new equities as described in (10). Newly issued equities are therefore (because of goods-market equilibrium) equal in amount to total savings which implies –
later on – that private savers will be just content with the supply of new equities by firms. Hence

\[ pE\dot{E}/p = I. \] (10)

The final equation describing firms’ behaviour is the capital accumulation equation which may be written

\[ \dot{K} = I/K. \] (11)

4. **Government (fiscal and monetary authority):**

In the government sector the total tax take, equation (12), is a constant function of profits plus interest payments in the hands of asset holders [alternatively we may assume that the ratio of taxes net of interest payments on government bonds to the value of the capital stock remains constant over time] i.e.

\[ T = \tau(pK + rB/p), \quad \text{[or] } t^n = (T - rB/p)/K = \text{const.}. \] (12)

Total government spending, equation (13), equals the total tax take net of interest payments on government bonds plus a term proportional (through \( \theta \)) to disequilibrium either in the labour market (as measured by \( V \)) or in the goods market (as measured by \( U \)). Thus government fiscal policy may aim at establishing full employment or full capacity use, since these two measures of labour and (implicit) goods market disequilibrium do not coincide in general. Note that this fiscal policy rule can be interpreted as (naively) Keynesian if \( \theta < 0 \) and as neoClassical otherwise (see Flaschel (1993), chapter 4 for further details). The additional \( \mu_2M/p \) term takes account of the fact that the government should continuously supply extra money in a growing economy and spend it on extra government purchases of goods, here to the extent described by the parameter \( \mu_2 \). Note that the most plausible parameter value for \( \mu_2 \) is \( n \) which characterizes the natural growth rate of the economy (implying that government debt will be zero in the steady state). Thus government spending \( G \) is given by

\[ G = T - rB/p + \theta(Q - \bar{Q})K + \mu_2M/p, \quad Q = V \quad \text{[or] } Q = U. \] (13)

In equation (14) government savings \((T - rB/p - G)\) are financed (if they are negative as is usually the case) by additional money and bond supplies \( \dot{M} \) and \( \dot{B} \). Thus

\[ S_g = T - rB/p - G, \quad [= -(\dot{M} + \dot{B})/p, \text{ see below}]. \] (14)
The monetary growth rule is specified in equation (15) and may adjust \((\mu_1 \neq 0)\) to the deviation of expected inflation from its steady state value. Hence we have

\[ \dot{M} = \mu = \mu_0 + \mu_1(\mu_0 - n - \pi). \]  

Since the magnitudes \(G, T, \text{and } \dot{M}\) are all determined by active policy rules, the amount of debt financing is determined passively in this model and has to be calculated from the government budget constraint (GBC) in equation (16), namely

\[ \dot{B} = pG + rB - pT - \dot{M} \quad [= (\mu_2 - \mu)M + \theta(Q - \bar{Q})pK]. \]  

Finally it should be pointed out that we assume that the government bonds are of the fixed-price variety (with price \(p_B = 1\)) and varying nominal interest parameter \(r\) per bond. In assuming this type of bond we follow Sargent (1987, p12), in viewing this asset essentially as a savings deposit.

5. **Equilibrium conditions (asset-markets):**

We have in this model the usual LM-equilibrium of Keynesian models which by the wealth constraint of asset holders and the perfect substitutibility assumption between bonds and equities implies that the other asset markets clear as well. Hence we write equation (17),

\[ \begin{align*}
M &= M^d = h_1pY + h_2pK(1 - \tau)(\bar{r} - r), \\
B &= B^d, \\
E &= E^d.
\end{align*} \]  

Here \(h_1, h_2\) are assumed to be positive constraints. As previously stated investment is assumed in this model to be entirely financed by equities issued by firms. The asset structure that is available to capitalists therefore now consists of outside money, government bonds and equities (see Sargent (1987, p12) for the same starting point ). Equities and bonds are assumed to be perfect substitutes in the eyes of asset holders which represents the most basic assumption that can be made in this context. We assume in this model that there are no planned retained earnings of firms which means that all expected profits \(\rho pK\) are paid out to equity owners in each period. Calculated after taxes equity owners therefore actually receive \((1 - \tau)\rho pK/E\) as return per unit of equity. The price of equities is determined by the above perfect substitute assumption which means that the actual rate of return on equities per unit of money is given by \((1 - \tau)\rho pK/(\rho EE)\). Under the perfect substitute assumption this
must be equal to \((1 - \tau)r - \pi\), the real rate of interest after taxes. This economic relationship yields equation (18) of the model,

\[
p_{EE} = (1 - \tau)pK/((1 - \tau)r - \pi).
\]  \( (18) \)

Equation (19) states that asset holders will voluntarily accept the additional supply of money and bonds and adjust their thereby changed portfolios only in the 'subsequent period'. Due to the resulting implied equality \( S = S_p + S_g = pE\dot{E}^d \), (see (5),(14)), we obtain from (9) the equation \( \dot{E} = \dot{E}^d \), i.e. general consistency with respect to flows\(^4\) (besides the general consistency for stocks (17)). Hence we have

\[
\dot{M} = \dot{M}^d, \quad \dot{B} = \dot{B}^d, \quad \dot{E} = \dot{E}^d.
\]  \( (19) \)

6. **Equilibrium condition (goods-market):**

In the goods market equality of investment and savings (IS – equilibrium) is given by

\[
S = p_{EE}E^d = S_p + S_g = Y - \delta K - C - G = I = pE\dot{E}.
\]  \( (20) \)

Thus our model has output determined by effective demand (via IS – equilibrium) and allows for underutilised capital (besides underutilised labour). The utilization rate of the capital stock is here considered as the essential equilibrating variable which adjusts immediately to clear the market for goods, while the price level will respond to changes in capacity utilization only with a time delay (given by the reciprocal value of the adjustment speed \( \beta_p \) of prices \( p \) see eqn. 22 below). Our modeling procedure thus assumes that supplies (output) respond faster than prices (the price level), in the strict sense that the former variable is assumed as a statically endogenous one, i.e., capable of performing jumps to restore temporary equilibrium, while the latter assumed as dynamically endogenous, i.e., changes continuously in time according to its law of motion.

7. **Wage-Price-Sector (adjustment equations):**

Finally consider the wage price sector. In equation (21) wage inflation is proportional to deviations of the rate of employment from its NAIRU - based level \( \dot{V} \). In addition wage inflation is assumed to be influenced by the observed actual rate of inflation as well as the expected rate of inflation. The rate of price inflation, equation (22), besides being driven by

\(^4\)Note here again that this 'simple' assumption has strong implications for the ability of the government to influence the pace of capital accumulation. Nevertheless, we shall not dispense with this standard assumption of continuous-time macrodynamic theory in this paper, but shall leave its detailed reconsideration for future investigations.
the rate of wage inflation and the expected rate of inflation is driven by a second type of
Phillips curve – besides the "proper" Phillips curve equation (21). This second Phillips curve
relates price inflation with deviations of actual capacity utilization \( U \) from the desired one \( \bar{U} \).
This desired rate of capacity utilization is exogenously given and it plays the same role as the
NAIRU rate \( V - \bar{V} \) in the many models of inflation theory that are based on such a concept
(or even on the NAR). Equation (23) describes the formation of inflationary expectations \( \pi \).
These expectations can be either purely backward looking (adaptive expectations) if \( \beta_{n_2} = 0 \)
or purely forward looking (towards the new steady state value \( \mu_0 - n \) of the rate of inflation)
if \( \beta_{\pi_1} = 0 \) holds (regressive expectations) – or they may be a combination of both. Thus
wage, price and inflationary expectations are driven by

\[
\begin{align*}
\dot{w} &= \beta_w (V - \bar{V}) + \kappa_w \dot{p} + (1 - \kappa_w) \pi, \\
\dot{p} &= \beta_p (U - \bar{U}) + \kappa_p \dot{w} + (1 - \kappa_p) \pi, \\
\dot{\pi} &= \beta_{\pi_1} (\dot{p} - \pi) + \beta_{n_2} (\mu_0 - n - \pi).
\end{align*}
\]

(21) (22) (23)

The above symmetric formulation of wage and price Phillips curves has been neglected
in the macroeconomic literature on inflation, due on the one hand to the identity that is
normally assumed between the rate of wage and of price inflation (based on the original
Samuelson/Solow assumption of a simple static markup theory of the price level) and due
on the other hand to the assumed validity of Okun’s Law according to which the utilization
of labor and capital are (always?) positively related. This need not be the case in the model
here which demands that we have to distinguish between the above two Phillips curves (and
their NAIRU’s) from now on. It would be interesting to have empirical observations on this
second type of curve in comparison to the many observations that exist for its money-wage
counterpart. We here stress once again that the above type of markup pricing behavior is
also (at least partly) present in this goods market reaction curve through the term \( \kappa_p \dot{w} \).

It is worth digressing at this point to compare Keynesian prototype with the Keynes
–Wicksell prototype we have recently developed in Chiarella and Flaschel (1995a). The Keynes
–Wicksell prototype has most of the features of our Keynesian prototype except that there is
disequilibrium in the goods market (\( I \neq S \)) and this disequilibrium feeds into the formation
of inflation (with a term \( \beta_p (I - S)/K \) in place of \( \beta_p (U - \bar{U}) \) in equation (22)). There is
one central weakness and problematic feature in the formulation of such a general Keynes-
Wicksell prototype model which indeed disqualifies it as a candidate for a truly Keynesian
dynamics – as (at least) its limit case \( \beta_p = \infty \) (augmented by a neoClassical production
function) is generally viewed and classified, see for example Sargent(1987), McCallum(1989).
This weakness is given by the fact that the firms of this model always operate at full capacity
\( Y = Y^p \) – which also remains true for the case of smooth factor substitution – as if Say’s
Law would hold true. Of course, there may be deficient or excess demand in this model
on the market for goods, but its effect solely is on the rate of inflation of this economy
(if \( \dot{K} = S/K \) holds), while there are in addition unplanned inventory fluctuations with no
further consequences in the case where \( \dot{K} = I/K \) holds, see Chiarella and Flaschel (1995a)
for details. It is furthermore simply assumed in such a Keynes-Wicksell prototype that
these inventory fluctuations stay within such bounds that their influence on the evolution
of quantities can be neglected. In the limit case $\beta P = \infty$ (so that $I = S$) of the Keynes-
Wicksell model, this unkeynesian feature of the model becomes even more pronounced, since
it then gives rise to a model with full capacity utilization (but unemployed labor – due to
nominal wage rigidities), where the Keynesian IS-LM equilibrium part is solved through
price and nominal interest rate adjustments and thereby adjusted to the predetermined
supply of commodities in each period. The model is then definitely following proposals
made by Friedman in the early seventies with respect to a monetarist reformulation of IS-
LM analysis and may therefore be characterized as being of Wicksell-Friedman rather than
of Keynes-Wicksell type.

These shortcommings of models in the Keynes – Wicksell vein are overcome by the Keynesian
demand oriented model we have formulated here with its implications for both the utilization
of capital as well as labor. A further justification for such a Keynesian reformulation is given
by the empirical fact that severe disequilibrium on the side of capitalist firms does not
so much show up in demand and supply imbalances for their products as in the Keynes-
Wicksell model, but in a severely underutilized capital stock – since production is relatively
easily adjusted to deficient aggregate demand. The disequilibrium measure used in Keynes-
Wicksell models (via the term $\beta P(I - S)/K$ in the price adjustment equation) is therefore
only appropriate when firms always operate at full capacity (as is assumed to be the case in
this model type), but it is very implausible in a model that allows for underutilized capital
– as a Keynesian model should do.

Our Keynesian prototype therefore starts from IS-equilibrium – which makes the measure
of goods market disequilibrium of Keynes – Wicksell models completely irrelevant for the
discussion of the determinants of the rate of inflation. IS-disequilibrium can, however, be
integrated into the above ‘proper’ (though still very simple) prototype of a Keynesian dynam-
ics. This is done in Chiarella and Flaschel (1994) where it is seen that the Keynes-Wicksell
treatment of it is very misleading as far the ‘Keynes’ label in denoting this model type is
concerned.

Starting from the general Keynes-Wicksell model type just discussed the derivation of a
proper Keynesian monetary growth dynamics is in fact not very difficult and demanding so
that one may wonder why the above new Keynesian prototype model is not yet a standard
model of the macroeconomic literature of Keynesian dynamics. A partial explanation for
this fact is the wide-spread (and un-reflected) use of a neoClassical production function
in combination with the Classical postulate on the equality between the actual marginal
product of labor and the real wage in the conventional macroeconomic literature which may
be responsible for the bastard model of Keynesian dynamics so widely used in the literature.
The exception to this characterization of Keynesian dynamics is given by so-called fix-price
approaches where, however, the dominance of the so-called Keynesian regime – as described
by the above model – is not clearly established and where there is no far-reaching and
generally accepted prototype of monetary growth available.
Textbook and other presentations of the Keynesian model often treat the price level as a
statically endogenous variable (determined by the marginal wage cost rule). It is shown
in Chiararella and Flaschel (1994) that the Keynesian model – augmented by smooth factor
substitution – becomes a supply determined model when prices as well as quantities adjust
with infinite speed. Such a modification of the model should therefore be considered as
one which leads us back to a (Neo)Classical scenario and should consequently be excluded
from the set of proper, i.e., demand determined Keynesian models. In fact, the Keynesian
model with a perfectly flexible price level is basically identical to the same limit case of the
Keynes–Wicksell model where the price level and the nominal rate of interest adjust aggregate
demand to the predetermined level of aggregate supply at each moment in time. One may
nevertheless insist that prices and quantities should be treated in a more symmetrical way
– and this issue is taken up in Chiararella and Flaschel (1995b) where it is assumed that both
the adjustment of output as well as that of prices occur with a finite speed (which may be
very different nevertheless). \(^5\)

2.2 The Dynamic Structure and its Steady–State

As far as the mathematical analysis of the model is concerned we shall mostly concentrate
on a modification of it which assumes that taxes net of interest payments per value unit of
capital are a given magnitude. The tax–rate \(\tau\) can then be suppressed, in particular in the
investment equation and in the perfect–substitute assumption.

Before we start the analysis of this special case \(t^n = (T - kB/p)/K = \text{const.}\), let us briefly
present the general case of an endogenous determination of this ratio and thus the existence
of a feedback mechanism (still a simple one!) of government debt accumulation \(\dot{B}\) on the
rest of the dynamics. From equations (1) – (23) we obtain the following autonomous five–
dimensional dynamical system\(^6\) in the variables \(\omega = \omega/p, \lambda = \lambda/K, m = m/(pK), \pi\) and
\(b = B/(pK)\): \(^7\)

\[
\begin{align*}
\dot{\omega} &= \kappa[(1 - \kappa_\pi)\beta_\pi X^\omega + (\kappa_\pi - 1)\beta_\pi X^\pi] \quad (24) \\
\dot{\lambda} &= n - s(\cdot) = -i_1(\cdot) - i_2(\cdot) \quad (25) \\
\dot{\pi} &= \beta_\pi [\beta_\pi X^\pi + \kappa_\beta \beta_w X^w] + \dot{\lambda} \quad (26) \\
\dot{\pi} &= \beta_\pi [\beta_\pi X^\pi + \kappa_\beta \beta_w X^w] + \beta_\pi (\mu_0 - n - \pi) \quad (27) \\
\dot{b} &= (\mu_2 - \mu) m - (\pi + n)b - (\kappa(\beta_\pi X^\pi + \kappa_\beta \beta_w X^w) - \dot{\lambda})b + \theta X^w \quad (28)
\end{align*}
\]

\(^5\) Note here however that ‘output’ \(Y\) is in fact ‘the rate of output’ or ‘output speed’ in a continuous time
model and is therefore quite different in dimensionality as compared to the price level \(p\). Such a speed
variable may change instantaneously (jump) without leading to jumps in the level of output (and the level of
prices). Nevertheless a Keynesian model should also be capable of allowing for finite adjustments in output
speed (see Chiararella and Flaschel (1994)).

\(^6\) Note here that fiscal policy may alternatively also be directed towards an elimination of the disequilibrium
\(X^p = U - \dot{U}\) in the place of the \(V - \dot{V}\)-measure employed in the following formula.

\(^7\) Making use of the abbreviation \(\kappa = (1 - \kappa_\pi \kappa_\beta)^{-1}\).
where we employ again the abbreviations \( \bar{r} = r_0 \) again:

\[
\begin{align*}
\rho &= y - \delta - \omega l^d, \quad l^d = L^d/K = y/x \quad (y \text{ not const.}), \\
X^w &= l^d/l - \bar{V} = y/(xl) - \bar{V}, \quad XP = U - \bar{U} = y/y^p - \bar{U}, \\
r &= r_0 + (h_1 y - m_0)/(h_2(1 - \tau)) \\
t &= T/K = \tau(\rho + rb), \quad t^n = t - rb, \\
g &= t^n + \theta X^w + \mu_2 m, \quad \mu = \mu_0 + \mu_1(\mu_0 - n - \pi) \\
s(\cdot) &= \dot{K} = s_c(\rho - t^n) - (g - t^n) = I/K, \\
i_1(\cdot) &= i_1((1 - \tau)\rho - (1 - \tau)r + \pi), \quad i_2(\cdot) = i_2 X^p,
\end{align*}
\]

and the following IS-LM determination of actual output per capital \( y = Y/K \) (\( \neq y^p = Y^p/K \) in general):

\[
\begin{align*}
s(\cdot) &= s_c(1 - \tau)(\rho + rb) - \theta X^w - \mu_2 m = \\
i(\cdot) &= i_1((1 - \tau)y(1 - u) - \delta - (1 - \tau)r + \pi) + i_2(U - \bar{U}) + n
\end{align*}
\]

where \( u = \omega/x \) is the share of wages in gross national income and where \( r \) is a function of \( m, y \) as given above. We note that this (linear) equation can be solved explicitly for output per capital \( y \) (this determination of \( y \) is discussed in more detail in section 3). It is not difficult to establish that the unique steady-state point of the five-dimensional dynamical system (24)-(28) is given by

\[
\begin{align*}
y_0 &= y^p, \quad l_0 = y_0/x = l^d_0, \\
m_0 &= h_1 y_0, \quad \pi_0 = \mu_0 - n, \\
r_0 &= \left[ 1 + \frac{(\mu_2 - \mu_0)}{\mu_0} m_0 \right]^{-1} \cdot \left[ n + \frac{\mu_2 m_0 + s_c \pi_0}{s_c(1 - \tau)} \right], \\
\omega &= \frac{1}{l_0} \left[ y_0 + \frac{\pi_0}{(1 - \tau)} - \delta - r_0 \right].
\end{align*}
\]

Let us now assume \( t^n = t - rb = \text{const} \) and remove the parameter \( \tau \) from the equations of the model (since taxes are now lump-sum).\(^8\) Furthermore, we set \( U = \bar{V} = 1 \) for notational simplicity.\(^9\) This gives a four-dimensional dynamical in system \( \omega, l, m, \pi \) – with an appended \( \bar{b} \)-dynamics – since the influence of \( rb \) on \( s(\cdot) \) and \( g \) (and thus on \( y \)) is thereby suppressed. These dynamics will be investigated with respect to local as well as global stability properties for a variety of subcases.

---

\(^8\) \((1 - \tau)(\rho + rb) = y(1 - u) - \delta - t^n.\)

\(^9\) through an appropriate redefinition of the sizes of the parameters \( x, \beta_w \) and \( y^p, \beta_p.\)
We assume in this analysis that the implied evolution for $b$ will remain a bounded one. This can be expected to hold at least for the case of an asymptotically stable steady–state of the four–dimensional subsystem $\dot{\omega}, \dot{i}, \dot{m}, \dot{\pi}$, since $X^p, X^w, \dot{l}$ all tend to zero in this case, so that the dynamics will then be dominated by

$$\dot{b} = (\mu_2 - \mu)m - (\pi + n)b$$

(the steady–state value of $b$ is $b_0 = (\mu_2/\mu_0 - 1)m_0$). Note here also that the coefficient $-(\pi + n)$ of this differential equation characterizes the entry $J_{55}$ of the Jacobian of the full dynamics and thus gives the fifth eigenvalue of this Jacobian in the present disconnected situation. The local stability properties of the following system are therefore augmented by the additional $b$–equation and 'its' eigenvalue in a straightforward way.

In the present case $t^n = \text{const.}$ there is again a unique steady–state configuration for the dynamics (24) – (28) with $\omega_0, l_0, m_0 \neq 0, (\vec{r} = r_0, \vec{U} = \vec{V} = 1)$ given by:

\begin{align*}
y_0 & = y^p, \quad l_0 = y_0/x = l_0^d, \\
m_0 & = h_1y_0, \quad \pi_0 = \mu_0 - n, \\
\rho_0 & = t^n + \frac{n + \mu_2 m_0}{s_c}, \\
\omega_0 & = (y_0 - \delta - \rho_0)/l_0^d, \\
r_0 & = \rho_0 + \mu_0 - n.
\end{align*}

We assume that the parameters of the model are chosen such that the last three steady state values are all positive. All following analysis will be confined to stability investigations of or around this steady–state of the given model.

Summarizing, the system (with $t^n = \text{const.}$) that we shall investigate reads:

\begin{align*}
\dot{\omega} & = \kappa [1 - \kappa_p] \beta_u(y/(xl) - 1) + (\kappa_w - 1) \beta_p(y/y^p - 1)], \quad (29) \\
\dot{i} & = n - s_c(y(1 - \omega/x) - \delta - t^n) + \theta(y/(xl) - 1) + \mu_2 m, \quad (30) \\
\dot{m} & = \mu_0 + \mu_1 (\mu_0 - n - \pi) - \pi - n, \\
& \quad -\kappa [\beta_p(y/y^p - 1) + \kappa_p \beta_u(y/(xl) - 1)] + \dot{i}, \quad (31) \\
\dot{\pi} & = \beta_\pi \kappa [\beta_p(y/y^p - 1) + \kappa_p \beta_u(y/(xl) - 1)] + \beta_\pi (\mu_0 - n - \pi), \quad (32)
\end{align*}

where $y$ is given by the solution of $(u = \omega/x, x = \text{const})$:

\begin{align*}
s(\cdot) & = s_c(y(1 - u) - \delta - t^n) - \theta(y/(xl) - 1) - \mu_2 m = \\
i(\cdot) & = i_1(y(1 - u) - \delta - (r_0 + (h_1 y - m)/h_2 + \pi)) + i_2(y/y^p - 1) + n, \quad (33)
\end{align*}
from which we determine \( y \) as a function of \( u, l, m \) and \( \pi \) i.e.

\[
y = \frac{[s_c(\delta + \epsilon n) - i_1(\delta + r_0) - i_n + n - \theta] + (\mu_2 + i_1/h_2)m + i_1 \pi}{(s_c - i_1)(1 - u) - \theta/(xI) + i_1h_1/h_2 - i_2/yP}.
\]  (34)

Note here that we have set the two parameters \( \bar{U}, \bar{V} \) both equal to 1 in this presentation of the dynamics in order to simplify the expressions and investigations of the model in the following sections.\(^{10}\)

2.3 The Model’s Nonlinear Structure

An important part of our analysis of the dynamic behaviour of the four dimensional system (29) – (32) will be the endogenous bounded fluctuations (typically limit cycles) to which it can give rise when the steady-state is locally unstable. We know from the theory of differential equations that such fluctuations are due to nonlinearities in the dynamical system (29) – (32). Nonlinearities in our Keynesian system can occur in a number of ways. They can be introduced via the underlying economic behavioural relationships such as the investment function and speeds of adjustment to name two that we shall consider later. However even when all the economic behavioural relationships are assumed to be linear, our Keynesian prototype has a naturally occurring nonlinear structure arising from two sources. The first is the nonlinear dependence of \( y \) on \( \omega, l, m \) and \( \pi \) in equation (34) which arose via the IS equation. This relationship is of a quotient type. The second nonlinearity arises due to the fact that the differential equations for \( \omega, l \) and \( m \) are expressed in terms of growth rates \( \bar{\omega}, \bar{l} \) and \( \bar{m} \). In the absence of any other nonlinearity this second nonlinearity would give rise to generalised Volterra predator–prey dynamics (see e.g. Peschel and Mende (1986)) which are known to be capable of generating limit cycle behaviour. In our subsequent analysis we shall first analyse the system dynamics with the naturally occurring nonlinearities, and only introduce nonlinear economic behavioural relationships in a second phase of analysis.

In the next section we shall briefly investigate the IS-LM subsector of the model, in particular in the case where the nominal rate of interest and the expected rate of inflation are given magnitudes. There exist three typical situations for the dependence of the point of effective demand \( y \) on income distribution. These three cases give rise to three different phase diagrams when we consider the simple Goodwinian interaction between real wage formation and capital accumulation in section 4, phase plots which no longer need to be very close to

\(^{10}\)One essential difference of these dynamics to the dynamics of the general Keynes-Wicksell model is that the \( \bar{p} \)-dynamics are now more roundabout, since the IS-part of the model does not determine the rate of inflation immediately, but instead determines the equilibrium output ratio \( y \) which then via the rate of capacity utilization determines the rate of inflation. Note here that low degrees of capacity utilization are much more plausible than large \( I < S \)-discrepancies as a general description of goods-market disequilibria. Note also that goods-market disequilibria have now – and must have – an impact on investment behavior. This is another essential difference between Keynes-Wicksell and Keynesian models.
the original structurally unstable center type dynamics of the Goodwin model. The Rose extension of the real wage dynamics is then considered in section 5, particularly in their potential to generate limit cycles from the Goodwin overshooting mechanism still present in cases 2 and 3. Section 6 extends this dynamics to dimension 3 by again including the money market and interest rate phenomena. We shall see there that the Keynes effect \( y_m > 0 \) will play a prominent role in the return to asymptotic stability that can there be observed in particular. This increase in the stability potential is, however, again reduced or overthrown when adaptively formed expectations are added to the model (in section 7) – due to the destabilizing potential of the Mundell effect \( y_\pi > 0 \).
3 Comparative statics: The IS–LM subsector

As already noted, the temporary equilibrium part of the model determines actual output per capital $y$ (and the rate of interest $r$) at each moment in time by means of equation (34). We shall discuss in this section the function $y(u, m, \pi)$, $u = \omega/x$, $m = M/(pK)$ that is defined by eq. (34) for the special case $\theta = 0$, (i.e., $g = t^n + \mu_2 m$, $t^n = \text{const}$) and $\mu_1 = 0$. This case represents the situation where a medium run oriented fiscal and monetary policy is absent from the model, leaving only the steady influences of $t^n$ and $\mu_0, \mu_2$. The equation for the determination of actual output then reads:

$$y = \frac{(s_c(\delta + t^n) - i_1(\delta + r_0) - i_2 + n) + (\mu_2 + i_1/h_2)m + i_1\pi}{(s_c - i_1)(1 - u) + i_1 h_1/h_2 - i_2/y_p}. \quad (35)$$

The function $y(u, m, \pi)$ in equation (35) which is locally well-defined whenever the denominator is non-zero has the following expressions for its partial derivatives:

$$
\begin{align*}
    y_u &= \frac{\partial y}{\partial u} = \frac{(s_c - i_1)y}{(s_c - i_1)(1 - u) + i_1 h_1/h_2 - i_2/y_p}, \\
    y_m &= \frac{\partial y}{\partial m} = \frac{\mu_2 + i_1/h_2}{(s_c - i_1)(1 - u) + i_1 h_1/h_2 - i_2/y_p}, \\
    y_\pi &= \frac{\partial y}{\partial \pi} = \frac{i_1}{(s_c - i_1)(1 - u) + i_1 h_1/h_2 - i_2/y_p}.
\end{align*}
$$

The common denominator of these expressions shows the three different forces that act on output per capital $y$ via investment and savings behavior, i.e., the real-wage effect $(s_c - i_1)(1 - u)$, the capacity-effect $-i_2/y_p < 0$ and the interest rate effect $+i_1 h_1/h_2 > 0$. The sign of the real-wage effect depends on the relative sensitivity of savings and investment with regard to changes in the real wage. The sign of the denominator can be positive or negative, depending on the choice of $s_c, i_1, i_2$ and $h_{1,2}$ in particular. Two cases will be of particular importance in the following: $h_2 \approx \infty$ and $h_2 \approx 0$. In the second case, the interest rate effect dominates the denominator and makes it unambiguously positive, if $i_1, i_2 > 0$ stay within an economically meaningful range. In the case $h_2 \approx 0$ we then get:

**Proposition 1:** Assume that $y > 0, u \in (0, 1)$ is guaranteed and that $h_2$ is chosen sufficiently small. Then:

$$y_u \geq 0 \iff s_c \geq i_1, \quad y_m > 0, \quad y_\pi > 0.$$
Remarks:

1) The effect \( y_m > 0 \) is called a (normal) Keynes–effect in the literature, while \( y_\pi > 0 \) is the so-called (again normal) Mundell effect. Increasing money supply and inflationary expectations thus both increase economic activity in this case.

2) Note that the Keynes–effect and the Mundell–effect are reversed in sign if the denominator expression becomes negative, i.e., if \( s_c < i_1 \) and \( h_1 / h_2 \) sufficiently small. Such a perverse Keynes–effect will, however, be small in size in general. Note also that \( y_u \) will be generally positive then.

3) Note finally, that \( \rho_u = y_u(1-u) - y \) can be negative (for \( 1-u > 0 \)) even if \( y_u > 0 \) holds.\(^{11}\) Most of our following investigations will rely on the prevalence of the conditions \( \rho_u < 0 \), \( y_m > 0 \) and \( y_\pi > 0 \), i.e., on a normal functioning of the profitability effect of real wage increases, the Keynes–effect and the Mundell–effect (but with \( y_u \geq 0 \)).

Let us now consider the case \( r = r_0 \ (h_2 = \infty) \) and in addition we assume again \( \mu_2 = 0 \), and \( \pi = \mu_0 - n \ (\beta = \infty) \) as side–conditions in order to obtain two–dimensional dynamics (in \( \omega, \ell^s \)) for this special case which is independent of the rest of model. In this case we can easily (by use of the steady–state relationships) reduce (35) to:

\[
y = \frac{(s_c - i_1)(1 - u_0)y^p - i_2y^p}{(s_c - i_1)(1 - u)y^p - i_2} = \frac{N}{D} \cdot y^p
\quad (36)\]

For the denominator \( D \) of this expression we can have one of the four situations depicted in figure 1 (cases 1, 2a,b: \( s_c > i_1 \), 3: \( s_c < i_1 \)) if \( u_0 \in (0,1) \) holds.

\(^{11}\)\( \rho_u \) will indeed be negative as long as \( y_m, y_\pi > 0 \) and \( i_2/y^p < h_1/h_2 \) holds, i.e. for all \( i_2 \) that are sufficiently small.
Figure 1:
The denominator in the effective demand function (36).

Figure 2a: Effective demand in case 1 ($\rho' > 0$) and case 3 ($\rho' < 0$)
- A too weak capacity effect (case 1) or $s_c < i_1$ (case 3)

Figure 2b: Effective demand in case 2a and case 2b ($\rho' < 0$)
- A strong capacity effect (relative to $s_c - i_1 > 0$)

It is clear from (36) that always we have $D = -i_2$ at $u = 1$. The denominator $D$ is thus positive in a neighborhood of the steady state value $u_0$ only in case 1 where we have $s_c - i_1 > 0$ and sufficiently large relative to the size of $i_2$. The benchmark case between case 2 and 3,
finally, is given by a horizontal line \((s_c - i_1 = 0)\) that passes through \(-i_2\). This figure implies for the shape of the function \(y(u)\) the two (four) possibilities displayed in figures 2a, b.\(^{12}\)

Furthermore, \(\rho(u) = y(u)(1 - u) - \delta\) is a decreasing function of \(u\) – and thus of an expected type – in cases 2 and 3. This is obvious for cases 2a, b and follows for case 3 from

\[
y(1 - u) = \frac{i_2 + (i_1 - s_c)(1 - u_0)}{i_2/(1 - u) + (i_1 - s_c)}
\]

since the denominator of this expression is positive in case 3.

Summarizing, we thus have in the present context

\[y_m = 0, y_\pi = 0\] and \(\rho'(u) < 0\)

in these two cases, while case 1 behaves perversely insofar as we then have a positive impact of the share of wages \(u\) on output \(y\) as well as on the rate of profit \(\rho\) (by a similar reasoning as in the case 3 we have just considered). It is to be expected that we will get a strange type of dynamics in this latter case.

We furthermore note that \(\rho(u)\) will be constant in the case \(i_2 = 0\) which should be considered a problematic situation, and that in the case \(i_1 = 0\) we have only the capacity effect operating in the investment function which nevertheless gives a meaningful situation (of type 1 or 2a, b).

Observe finally that \(\omega_0\) will be positive in the present special case if the data of the model fulfill

\[s_c(y^p - \delta - \tau^n) - n > 0.
\]

The steady-state value of \(\omega_0\) is then determined by

\[s_c(y^p - \delta - \omega_0 y^p/x - \tau^n) = n
\]

which implies that \(\rho_0 = y^p - \delta - \omega_0 l_0\) must be positive as well \((n > 0)\). Due to

\[\rho_0 = y^p(1 - u_0) - \delta
\]

we then get \(u_0 < 1\) (and \(u_0 = \omega_0/x > 0\)). This justifies our assumption \(u_0 \in (0, 1)\) made in the above figures. Finally, since \(r_0\) should also be positive, one has to assume that \(\rho_0 + \mu_0 - n\) is positive which is justified for \(\mu_0\) sufficiently close to \(n\), the natural rate of growth.

\(^{12}\)In the cases 3 and 2b the delimiting vertical line at \(A_1, A_{2a}\) must run instead through the points \(A_3, A_{2b}\), respectively. This enlarges the relevant phase space in an obvious way.
4 Variants of Goodwin’s growth cycle case

In this section we consider a special case of our general Keynesian model which allows us to focus on the dynamics of the real sector. These dynamics turn out to be of the Goodwin growth cycle type. However our analysis shows that a much more diversified outcome of the Goodwin(1967) profit squeeze mechanism comes about in the Keynesian version of real growth which in one case bears no resemblance with the overshooting mechanism of the original growth cycle model.

The construction of a real subsector of the model which is dynamically independent of the rest of the model is performed by making the following assumptions: (i) \( r = r_0 \) (the ‘liquidity trap’ at the steady-state rate of interest \( r_0 \)), and (ii) \( \pi = \mu_0 - n \) (infinitely fast regressive expectations). These two assumptions imply that investment per capital only depends on the wage share \( u = \omega/x \) and on the rate of capacity utilization \( U = y/y^p \). The additional assumption (iii), \( t^n = t - rb = \text{const.} \) supplemented (at first) by \( \theta = 0, \mu_2 = 0 \) and \( \mu_1 = 0 \), imply that, economic policy is here treated in a very simple way i.e. \( g = t^n \), \( \mu = \mu_0 \). The further assumption (iv), \( \kappa_w = 1 \) removes the Rose-effect (the influence of the theory of inflation on the dynamics of real wages) from the dynamics of the real sector.

The resulting model will have Keynesian demand features and is a Goodwin-type dynamical system of the kind \( u = \omega/x \):

\[
\dot{u} = \kappa (1 - \kappa_p) \beta_w(y(u)/(x_l) - 1), \tag{37}
\]

\[
\dot{l} = n - s_c(\rho(u) - t^n), \tag{38}
\]

with \( \rho(u) = y(u)(1 - u) - \delta \) where \( y \) is determined by eq. (36).

Note here that we no longer have \( \dot{V} = -\dot{l} \) for the rate of employment \( V = l^d/l \), since \( l^d = y/x \) is no longer a given magnitude but dependent on \( y \). The use of Goodwin’s variables \( u, V \) is therefore no longer a straightforward matter here. Since the original Goodwin growth cycle is structurally unstable we may expect that it will be changed in various directions depending on whether case 1, 2 or 3 of the preceding section is considered as determining the behavior of \( y(u) \) and \( \rho(u) \).

The Jacobian of the dynamics (37), (38) at the steady-state is given by

\[
J = \begin{pmatrix}
\beta_w y/(x_l) & -\beta_w y/(x_l^2) \\
-s_c(1/l) & 0
\end{pmatrix}
\]

For the three cases considered in the preceding section we thereby get for the above dynamics near the steady state:

**Proposition 2:** System (37), (38) is described locally by a saddle-path dynamics in case 1
(y' > 0, ρ' > 0), by a stable node or focus in case 2 (y' < 0, ρ' < 0), and by an unstable node or focus in the case 3 (y' > 0, ρ' < 0).

Proof: Evaluating the above Jacobian in these three situations gives:

Case 1: \( J = \begin{pmatrix} + & - \\ - & 0 \end{pmatrix} \), Case 2: \( J = \begin{pmatrix} - & - \\ + & 0 \end{pmatrix} \), Case 3: \( J = \begin{pmatrix} + & - \\ + & 0 \end{pmatrix} \).

These three representations immediately imply the three assertions of the proposition by means of the usual characterization of local phase plots via the determinant and the trace of the Jacobian.

Remark: An appropriate application of Olech's theorem, see Flaschel (1984) for details, should here imply that the last two stability characterizations also hold in the large, i.e. in the positive orthant \( \mathbb{R}_+ \).

The following proposition shows that the cases 2, 3 give rise to a Goodwin type dynamics if the wage adjustment parameter is chosen sufficiently small:

Proposition 3: Cases 2, 3 will exhibit a cyclical type of dynamics iff

\[
β_w < -4 \cdot \frac{s_e ρ'(u)u}{(y'(u)u/y)^2} \quad (>0).
\]

Proof: A straightforward implication of the calculation of the discriminant

\[
4\Delta = (\text{trace } J)^2 - 4\det J
\]

which is exactly negative – and thus gives rise to complex roots – when the above condition is fulfilled.

The phase diagrams of the three types of dynamics just considered are displayed in figures 3a, b, c (see figures 1, 2 in the preceding section with respect to the boundary values \( A_i, i = 1, 2, 3 \)).
Figure 3a:
Case 1: The 'paradise' case ($y'(u), \rho'(u) > 0$).
A saddlepoint.

Figure 3b:
Case 2: The 'orthodox' case ($y'(u), \rho'(u) < 0$).
Stable node or focus.
We note here without proof (see Chiarella and Flaschel (1994) for details) that the mass purchasing power argument is valid in the cases 1, 2a,b in the sense that an increase in the real wage will ceteris paribus increase excess demand

$$\Delta y^d = y^d - y = (i_1 - s_c)(1 - u)y + i_2 y p + \text{const},$$

which leads to an increase in effective demand $y$ in the first case and to a decrease of it in cases 2a,b. The mass purchasing power argument is therefore only supported by case 1, when the goods market reaction is taken into account. In case 3, finally, we have that the cost effect of real wage increases outweighs the immediate mass purchasing power effect, so that there is excess supply created initially. Taking again into account the market reaction to this disequilibrium situation, however, here nevertheless gives that effective demand is increased by the real wage increase.

It is possible to tailor the stable dynamics of case 2 further so that trajectories stay within a certain corridor around the steady state. To this end, assume that the adjustment speed of money wages with respect to $V(= y(u)/(xI))$, labor market disequilibrium, is $\beta_u(V)$ which has the form shown in figure 4.
Figure 4: The parameter $\beta_w(V)$, the speed of wage adjustment.

The value of $l$ which together with $y(u)$ satisfies $l^d/l = y(u)/(x l) = V_a$ or $= V_b$ is given by $l_a = y(u)/(x V_a)$ or $l_b = y(u)/(x V_b)$. Along these curves the rate of change $\dot{u}$ will be $-\infty$ or $+\infty$, respectively. We thereby get for case 2 the restricted phase diagram shown in figure 5.

Figure 5:
A region of global stability for case 2.

Since $V_a < 1$ and $V_b > 1$ due to our normalization of the value of $\dot{V}$ the employment rate will stay between $0 < V_a$ and $V_b < V_{max}$. The locally asymptotically stable situation of
case 2 can thereby easily be extended to an economically meaningful domain around the level '1' of the 'natural' rate of employment. A similar construction is not possible in the other cases considered here which will be further discussed in section 5 when the Rose real wage mechanism has been added to the dynamics. It should also be noted that this model still lacks a mechanism which keeps the evolution of the share of wages \( u \) below 1. As far as case 3 is concerned we note that all three curves \( \dot{u} = 0 \), \( l_a \) and \( l_b \) are increasing, i.e., the horizontal arrows of the above figure are then pointing outside instead of to the inside of the compact domain depicted in the above figure. The instability of case 3 is therefore significantly increased if wages become more and more flexible the further the rate of employment departs from its steady state value 1, leading here to a collapse of the dynamics in finite time.\(^{13}\)

We here add some brief observations on the case where an active fiscal policy is added to the above dynamics (to counteract the cycle and the occurrence of periods of under- (and over-) employment). Let us therefore now assume \( g = t^n + \theta(l^d/l - 1) = \theta(y/(xl) - 1) \) but still set \( \mu_2 = 0 \). The change that is thereby induced in eq. (38) is obvious and it amounts to the addition of the following matrix \( F \) to the Jacobian \( J \) we have considered above [as far as the local stability analysis of the steady-state is concerned].

\[
F = \begin{pmatrix}
0 & 0 \\
\frac{s_c \theta y'(u)}{x} & -s_c \theta
\end{pmatrix} = s_c \theta \begin{pmatrix}
0 & 0 \\
\pm & -
\end{pmatrix}
\]

Here, the \( \pm \) term refers to case 2 (\(-\)) and case 3 (\(+\)), respectively. It is easily shown that det\((J + F)\) is equal to det\(J\), so that the addition of an active fiscal policy in the above way does not influence this stability characteristic at the steady state. However the trace \((J + F)\) is more negative than trace \(J\). Recall again that such a fiscal policy may be called 'Keynesian' if \( \theta < 0 \) holds, since it then implies increases in government expenditures for \( l^d/l < 1 \) (and decreases in the opposite case). We therefore get – the following somewhat unexpected result in a Keynesian monetary growth framework.

**Proposition 4:** Adding a Classical (Keynesian) fiscal policy rule to the Keynesian growth cycle dynamics (37), (38), in Case 2 or 3 makes the stable case 2 more stable (less stable) and stabilizes the unstable case 3 (destabilizes it further) if the Classical type of policy rule is exercised with sufficient strength.

The growth cycle subcase of our Keynesian monetary growth dynamics therefore still supports the superiority of Classical fiscal policy rules over Keynesian ones.\(^{14}\)

In the context of our Keynesian growth model – where both labor and capital can be under-utilized – there is also a further related target for an active fiscal policy: The rate of capacity utilization \( U = y/y^p \) instead of the rate of employment \( V = l^d/l = y/(xl) \). Replacing the above fiscal policy rule by

\[
g = t^n + \theta(y/y^p - 1),
\]

\(^{13}\)In Flaschel (1993, pp.166 ff.) case 3 (with \( i_1 = 0 \)) is made a viable dynamics via an appropriately delayed type of mark-up pricing and the assumption that \( \kappa_w \) will be less than 1 for \( u \) close to 1.

\(^{14}\)See Flaschel (1993, Ch.4) for a similar conclusion in a real Keynesian growth model.
gives in this case as matrix $F$

\[
F = \begin{pmatrix}
0 & 0 \\
\theta y'(u)/y & 0
\end{pmatrix} = \theta \begin{pmatrix}
0 & 0 \\
\pm & 0
\end{pmatrix},
\]

where now the ± term refers to case 2(+) and case 3(-). Since $J_{13}$ is always negative we therefore get in this case the proposition:

**Proposition 4':** A Classical policy can destabilize the steady state in the stable case 2 if exercised with sufficient strength, while a Keynesian one cannot. On the other hand, a Classical policy cannot stabilize the unstable case, while a Keynesian one may make the instability more pronounced (and will create saddlepath instability eventually).

Slight changes in the specification of fiscal policy rules can therefore lead to important qualitative changes in their stabilizing potential.
5 Rose-type employment cycle extensions

The three outcomes of section 4 are here further analyzed by including again the more complicated real wage mechanism of the Rose(1967) employment cycle in which the dynamics of the real wage also depends on the state of the market for goods. We shall find that either wage or price flexibility may now be destabilizing accompanied by stabilizing properties of the complementary wage or price adjustment mechanism. Such a scenario will again allow for Rose's(1967) nonlinear limit cycle analysis where, now however, the stabilizing and destabilizing factors may interchange their role with each other.

We therefore extend the dynamic feedback possibilities of the model in section 4 by allowing for \( \kappa_w < 1 \) while keeping all other assumptions of that section. The dynamical system for \( u, l \) then reads \((\rho(u) = y(u)(1 - u) - \delta)\):

\[
\begin{align*}
\dot{u} & = \kappa[(1 - \kappa_p)\beta_u(y(u)/(xl) - 1) - (1 - \kappa_w)\beta_p(y(u)/y^p - 1)], \\
\dot{l} & = n - s_c(\rho(u) - t^n).
\end{align*}
\]  
(39)  
(40)

From equation (40) we can determine the steady-state value of \( u \) (via \( \dot{l} = 0 \)) as given by

\[
y(u_0)(1 - u_0) = \delta + t^n + n/s_c.
\]

With respect to this value of \( u \) we have by assumption \( r_0 = \rho(u_0) + \mu_0 - n \) and therefore \( n = s(\cdot) = i_1(\cdot) + i_2(\cdot) + n \) with \( i_1(\cdot) = 0 \), i.e., \( i_2(\cdot) = 0 \) which implies \( y(u_0) = y^p \), i.e. we have full capacity utilization at the steady state value of the share of wages \( u \). By equation (39) we then get for the steady state (via \( \dot{u} = 0 \)): \( y^p/(xl_0) = 1 \) or \( l_0 = y^p/x \), as in the Goodwin subcase.

The above extension of the Goodwin case implies for the entry \( J_{11} \) of the Jacobian \( J \) we considered in section 4,

\[
J_{11} = u\kappa[(1 - \kappa_p)\beta_u/(xl) - (1 - \kappa_w)\beta_p/y^p]y'(u)
\]

while all other elements in the matrix \( J \) remain unchanged. This immediately implies the following proposition:

Proposition 5:

1. Increasing price-flexibility \( \beta_p \) in the Keynesian growth cycle dynamics destabilizes case 2 \((y', \rho' < 0)\) and stabilizes case 3 \((y' > 0, \rho' < 0)\) as far as the local asymptotic stability of the steady state is concerned.

2. Just the opposite holds for increases in wage-flexibility \( \beta_w \).
3. The steady state of case 1 can be made an asymptotically stable one if price-flexibility is increased sufficiently\(^{15}\) and if a Classical policy rule of the type \(\theta(y/y^p - 1)\) is exercised with sufficient strength.

In sum, we get in this Keynesian context that either increasing price-flexibility or increasing wage-flexibility may be stabilizing, but not both. Which one will be stabilizing, depends on the sign of \((i_1 - s_c)\) — if \(i_2\) is sufficiently large, so that case 1 can be excluded — and therefore on the relative sensitivity of investment and savings to changes in the real wage. If equilibrium output \(y\) falls (rises) with a rising real wage, the stabilizing role falls on \(\beta_w\) (on \(\beta_p\)).

In order to obtain phase portraits for the cases 2 and 3\(^{16}\) we calculate the \(\dot{u} = 0\) and \(\dot{l} = 0\) isocline of these two cases first. For the \(\dot{l} = 0\) isocline we have as in section 4 (\(\theta = 0\)):

\[
\rho(u_0) = t^n + n/s_c
\]

which determines a vertical curve in \((u, l)\)-space. And for \(\dot{u} = 0\), we get by (39):

\[
l = \frac{1/x}{(1 - q)/y(u) + q/y^p},
\]

where \(q = \frac{1 - \kappa_p}{1 - \kappa_w} \beta_p \beta_w\). The \(\dot{u} = 0\) isocline is well-defined as long as \(U = y(u)/y^p > 1 - 1/q\) (\(q > 0\)) holds true, since the denominator of the above fraction is then positive (and only then). This domain includes the steady-state value \(U_0 = 1\), but the interval to the left of \(U_0\) that it includes becomes small if \(\beta_p\) becomes large with respect to \(\beta_w\). The derivative of this isocline reads:

\[
l'(u) = \frac{(1 - q)y'(u)}{[(1 - q)/y(u) + q/y^p]^2 xy^2},
\]

i.e. \(l(u)\) is strictly decreasing in case 2 \((y' < 0)\) if \(q < 1\) holds and strictly increasing if \(q > 1\). For \(q = 1\), the isocline will be strictly horizontal. Case 3 \((y' > 0)\) will give rise to results that are exactly opposite to that of case 2.

For \(q > 1\), we have \(1 - 1/q < 0\), i.e., the \(\dot{l} = 0\) isocline is well-defined for all \(u\) for which \(y(u)\) is well-defined (see the figures in section 3). Furthermore, its slope is of the same type as that of the \(\dot{l} = 0\) isocline in section 4, i.e., case 2 and case 3 have the same phase portraits as in the Goodwin case as long as \(\beta_p\) is so small that \(q < 1\) holds true. As \(q\) approaches 1 (from below) the \(\dot{u} = 0\) isoclines in these figures will become flatter and flatter and are strictly horizontal lines finally for \(q = 1\). This reestablishes the closed orbit structure of the original Goodwin model, since the model then reduces to

\[
\begin{align*}
\dot{u} &= \kappa(1 - \kappa_p)\beta_w y(u)\left[\frac{1}{x\dot{l}} - \frac{1}{y^p}\right], \\
\dot{l} &= n - s_c y(u)(1 - u) + \delta + t^n,
\end{align*}
\]

\(^{15}\)(\(\beta_p > \frac{1 - \kappa_p}{1 - \kappa_w} \beta_w = \frac{1 - \kappa_p}{1 - \kappa_w} \beta_w\) (\(q > 1\), see below).

\(^{16}\)Case 1 — the paradise case — is dismissed here, because it looks too exceptional from an empirical point of view.
which can be treated in the same way as the Goodwin–like versions considered by Chiarella and Flashel (1995) in their Keynes–Wicksell prototype model. In addition to proposition 4 we can now state that case 2 remains of the stable type we considered in the preceding section and that case 3 remains of the unstable type as long as $q < 1$ holds and that both cases undergo a bifurcation at $q = 1$ (passing the Goodwin closed orbit structure at this value of $q$) and then become explosive for $q > 1$ in case 2 and stable for $q > 1$ in case 3.

Case 2 therefore switches from stability to instability at $q = 1$ because of the Rose (or $\beta_p$) effect of price formation on real wages giving rise then to a phase portrait of the type shown in figure 6:

![Phase Portrait](image)

*Figure 6: Instability for case 2 via the Rose effect ($y' < 0, q > 0, \beta_p > \frac{1-\kappa_p}{\kappa_w} \beta_w$).*

By contrast, case 3 switches from instability to stability by passing through the Goodwin center case at $q = 1$, now giving rise to a phase portrait of the type displayed in figure 7 when $\beta_p$ is further increased.
Figure 7: Stability for case 3 via the Rose effect \((y' > 0, q > 1)\).

We claim, but do not prove here that this case should also be asymptotically stable from a global point of view, that is, each trajectory which starts between \(u^c\) and \(A_3\) (at a not too high level of \(l\)) should converge to the steady state \((u_0, l_0)\).

Let us now also consider the unstable cases: case 2 with \(q > 1\) and case 3 with \(q < 1\) from a global point of view and see to what extent these cases may be stabilized far-off the steady state by either a non-linearity in wage adjustment or in price adjustment.

Case 2 \((q > 1\) at the steady-state): Assume the same functional form for the adjustment speed of money wages as in the preceding section, see figure 4. Thus wage adjustment becomes infinitely fast as the rate of employment \(V = y/(x_l)\) approaches the level \(0 < V_a < 1\) from above and \(V_b > 1\) from below.

By contrast, the price adjustment speed is assumed as constant in the present situation and of such a size that \(\beta_p > \frac{(1-x_a)}{(1-x_a)}\beta_w(1)\)' holds. In place of the above \(\hat{u} = 0\) isocline we then get

\[
l = \frac{y(u)/x}{\beta^{-1}_w(\frac{1-x_a}{1-x_a} \beta_p(\frac{y(u)}{y_p} - 1))}
\]

which is well-defined as long as this is the case for \(y(u)\), see section 4, since \(V_a > 0\) has been assumed. Furthermore \(l_b(u) < l(u) < l_a(u)\) where \(l_a(u) = y(u)/(xV_a)\), \(l_b(u) = y(u)/(xV_b)\) as

\[\beta_w(V) = \frac{1-x_a}{1-x_a} \beta_w(1) > 1.\] Note that we now employ a nonlinear adjustment function \(\beta_w(V)\) in the place of the linear adjustment function \(\beta_w(V - 1)\).
in the preceding section. The effect of this nonlinearity in wage adjustment behavior is to modify the last phase diagram for the case 2 as shown in figure 8.

![Figure 8: Viability in the locally unstable case 2.](image)

Note that the denominator of the $\dot{u} = 0$ isocline is restricted to the interval $[V_a, V_b]$ while $y(u) \to \infty$ for $u \to A_2$ (from above) and $y(u) \to 0$ for $u \to \infty$. The existence of the above shaded compact domain allows in principle for the application of the Poincaré-Bendixson theorem. Note that this domain can extend to the right of $u = 1$ and thus needs in this case further restrictions to guarantee that the share of wages $u$ stays below $u = 1$. This can be done, for example, by means of an appropriate choice of $V_a$ and $l_a$. The result we have obtained above is summarized by

**Proposition 6:** All trajectories which start in the above depicted shaded domain approach a limit cycle that is contained in this domain and that contains the steady-state $u_0$, $l_0$ in its interior.

Wage flexibility may therefore be the appropriate cure in the case we have called the orthodox one ($y'(u) < 0$) for a fixed type of price flexibility that was chosen large enough to destabilize the steady-state locally.

**Case 3 ($q < 1$ at the steady-state):**
In principle, price flexibility should now be stabilizing if it increases sufficiently far off the steady state (for any given degree of wage-flexibility $\beta_u$), since it leads to $q > 1$ far-off the steady-state.

Let us assume by analogy to the above that the following situation holds for the price adjustment coefficient $\beta_p$ a function of the rate of capacity utilisation $U$ of the same form as
in figure 4 for $\beta_w$, but with $U_a, U_b$ replacing $V_a, V_b$. The second term on the right hand side of the differential equation (39) will thus have the general shape shown in figure 9.

\[ \beta_p(U)[U - 1] \]

\[ u_c \quad 1 \quad u_b \]

\[ U=y/y^p \]

Figure 9: A second Phillips-curve mechanism\textsuperscript{18}; the price adjustment function.

In this case we have for the $\dot{u} = 0$-isocline

\[ I = \frac{y(u)/x}{1 + \frac{1 - \kappa_p}{1 - \kappa_w} \frac{\beta_p(y(u)/y)^p}{\beta_w}} \]

which is only well-defined as long as

\[ \beta_w \frac{1 - \kappa_p}{1 - \kappa_w} < -\beta_p(\cdot) \]

holds true, see the above figure ($U > U^c, u > u_c$), since $y'(u) > 0$ holds in this case.

The phase portrait that is implied by this new situations is displayed in figure 10.

\textsuperscript{18}c = -\frac{1 - \kappa_p}{1 - \kappa_w} \beta_w.
This situation will again allow for an application of the Poincaré–Bendixson theorem, now leading to

**Proposition 7:** All trajectories which start in an appropriately chosen compact domain approach a limit cycle that is contained in this domain\(^\text{19}\).

Thus the unstable (mixed) case 3 of Goodwin–type is here stabilized by some sort of anti–Rose effect. Note that assuming only the boundary behavior of the price level at \(U_b\) is already sufficient for such a limit cycle result. The fact that the \(U_b\) boundary is vertical is new and different from the \(l_b\) situation, but of course \(l_b, l_a\) refer to infinite wage–flexibility which is of no help in the present situation.

In sum, we therefore have the results that \(\beta_w\)-flexibility cures \(\beta_p\)-instability in case 2 (\(y' < 0\)), while \(\beta_p\)-flexibility cures \(\beta_w\)-instability in case 3 (\(y' > 0\)). Thus, the kind of flexibility that is needed for the global stability of these dynamics depends on the \(i_t \geq s_e\)-regime. Whatever this particular regime may happen to be, one flexibility is always helpful while the other is then not helpful, but necessarily destabilizing.

Finally let us briefly look at case 1 also, to see what happens in this case when \(q\) passes through 1 and becomes larger than one. The relevant phase diagrams are displayed in figure 11. The dynamics of this case therefore remains of a saddlepoint type even if prices become

\(^{19}\text{Now on the basis of local instability of the steady–state caused by a sufficiently high wage flexibility, an instability that is then bounded by an ever-increasing price–flexibility far–off the steady state}\)
very flexible. This case therefore cannot be made asymptotically stable by increasing wage or price flexibility.

Figure 11:
The stability switch in the case 1.
6 Monetary growth cycles: The basic case

This section now dispenses with the assumption \( h_2 = \infty \) \( r \equiv r_0 \). It shows that the addition of (sufficient) interest rate flexibility may – due to the Keynes-effect – generate (local) asymptotic stability in the now integrated real and monetary growth dynamics of this section and may thus remove the Rose type limit cycles from it. We shall, on the other hand, see that the potential for local Hopf bifurcations for example with respect to price adjustment speeds, is significant in this three dimensional extension of the two-dimensional dynamics of preceding section. The interaction of real and monetary factors can therefore still give rise to cyclical evolutions for a given value of the parameter \( h_2 \) if price adjustment speeds become sufficiently high and may thus explain the joint occurrence of growth and fluctuations on the macrolevel. Note here that we still stick to the assumption \( \beta_{\pi_2} = \infty \) in this section so that the counteracting – destabilizing – role of the so-called Mundell effect still remains excluded from the present investigation of the dynamics of the model.

We have so far assumed with respect to money demand that it is given by

\[
m^d = h(y, r) = h_1 y + h_2(r_0 - r), \quad h_{1,2} > 0
\]

(with \( h_2 = \infty \) in the analysis of the last two section). LM equilibrium then gives rise to the expression:

\[
r = r_0 + \frac{h_1 y - m}{h_2} \quad (= r_0 \text{ if } h_2 = \infty).
\]

as the implied determination of the nominal rate of interest. Let us now go to the opposite extreme and consider values of the parameter \( h_2 \) which are sufficiently small (to be characterized below), so that the rate \( r \) may now become very sensitive to changes in the values of \( m \) or \( y \).

The investigations of the present section will all be local in nature (concerning a certain neighborhood of the steady state of the model solely). They therefore immediately apply also to all nonlinear types of money demand functions \( h(y, r) \) provided that the values of \( h, h_y, h_r \) at the steady state are given by \( h_1 y, h_1, -h_2 \) as in the above case of a linear money demand function. A special example of such a nonlinear demand function is given by \( y h(r - r_0), h(0) = h_1, h' < 0, h'(0) = -h_2/y_0 \). We here only note that an extension to such a nonlinear money demand function will become necessary if the global dynamics of this system is to be studied. In the presence of a Keynesian IS-LM block this may however create difficulties, since there may then be more than only one IS-LM equilibrium that fulfills equation (35) for a given set of dynamically endogenous and exogenous data. This creates the problem of which equilibrium to choose as the basis for the further development of the economy. We stress that this is not a problem in this paper as long as the assumption of a linear money demand as specified above is fulfilled.

We know from the analysis of section 3 that for \( h_2 \) chosen sufficiently small\(^{20}\) the partial

\(^{20}\)such that the denominator in eq. (35) is positive!
derivatives $y_u, y_m$ are all well-defined and fulfill\textsuperscript{21}

$$y_u \geq 0 \text{ iff } s_c \geq i_1, \quad y_m > 0,$$

(41)

for the function $y(u, m)$ that is here implied by equation (36) at least for a neighborhood of the steady state. We assume for the following that $h_2$ is chosen such that (41) holds true. Furthermore it is easy to show by means of equation (35) that the function $\rho(u, m) = y(u, m)(1 - u) - \delta$ must have a negative derivative $\rho'(u) = y'(u)(1 - u) - y(u)$ for all combinations of the $s_c, i_1$ parameters, if $h_2$ is sufficiently small,\textsuperscript{22} so that there is no longer a possibility for case 1 of the two preceding sections in the following.

Since we have only added interest–rate flexibility to the preceding section (but still retain the assumption of a given $\pi = \mu_0 - n$ plus $\theta = \mu_1 = 0$) we now have to treat the following three–dimensional dynamics:

\begin{align*}
\dot{u} &= \kappa[(1 - \kappa_2)\beta_w X^w + (\kappa_1 - 1)\beta_p X^p], \\
\dot{l} &= n - s(\cdot), \\
\dot{m} &= -\kappa[\beta_p X^p + \kappa_2 \beta_w X^w] + \dot{l},
\end{align*}

(42) (43) (44)

with

\begin{align*}
X^w &= y/(xl) - 1, \quad X^p = y/y^p - 1, \\
s(\cdot) &= s_c(\rho - t^n) - \mu_2 m,
\end{align*}

and where $y(u, m)$, $\rho = y - \delta - \omega y/x = y(1 - u) - \delta$, $u = \omega/x$ are given as discussed above.

The above dynamic system can be represented in a form as follows:

\begin{align*}
\dot{u} &= (\tilde{\beta}_w X^w - \tilde{\beta}_p X^p)u, \\
\dot{l} &= (const - s_c \rho + \mu_2 m)l, \\
\dot{m} &= -[\tilde{\beta}_p X^p + \tilde{\beta}_w X^w] + s_c \rho - \mu_2 m - const]m.
\end{align*}

where $\tilde{\beta}_w, \tilde{\beta}_p, \tilde{\beta}_w, \tilde{\beta}_p$ are all positive constants.

\textsuperscript{21}\textit{Note that this differs considerably from the $y_u$–characterizations given in sections 4,5 of this chapter. Note also that the final partial derivative $y_u$ ($> 0$ here) is not yet of importance in this section. Its sign represents the Mundell-effect of a rise of inflationary expectations on the level of effective demand, while $y_m > 0$ is an implication (or reformulation) of the Keynes-effect of IS-LM models.}

\textsuperscript{22}\textit{such that $i_1 h_1/h_2 - i_2 y^p > 0$ holds!}
In order to investigate the stability of these dynamics at the steady-state (as determined in section 2.2) we have to consider the eigenvalue structure of the Jacobian matrix.

\[
J = \begin{pmatrix}
(\tilde{\beta}_w X_u^w - \tilde{\beta}_p X_P^w)u & \tilde{\beta}_w X_l^w u & (\tilde{\beta}_w X_m^w - \tilde{\beta}_p X_m^w)u \\
-s_c \rho m \tilde{l} & 0 & -s_c \rho m \tilde{l}
\end{pmatrix}
\]

of the above dynamics at the steady state \( u = u_0, \tilde{l} = \tilde{l}_0, m = m_0 \). Since the parameter \( \mu_2 \) can be considered as small we shall neglect it in the following calculations of stability criteria.

**Lemma 1:** The determinant \( |J| \) of the above Jacobian \( J \) has the same sign as \(-y_m\) and is a linear function of the parameter \( \beta_w \) (as well as \( \beta_p \)).

**Proof:** Straightforward when the linear dependencies in the matrix \( J \) are taken into account appropriately.

The sign of the determinant of \( J \) thus depends on the sign of the Keynes-effect solely.

**Lemma 2:** The three leading principal minors of \( J \) read

\[
J_1 = \begin{vmatrix}
J_{22} & J_{23} \\
J_{32} & J_{33}
\end{vmatrix} = -s_c \lambda_1 \tilde{\nu}_m (1 - u) X_l^w,
\]

\[
J_2 = \begin{vmatrix}
J_{11} & J_{13} \\
J_{31} & J_{33}
\end{vmatrix} = s_c \lambda m (\tilde{\beta}_p \lambda P - \tilde{\beta}_w \lambda (x \lambda)) y_m,
\]

\[
J_3 = \begin{vmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{vmatrix} = s_c \tilde{\nu}_m X_l^w u \rho u, \quad \rho u = y_u (1 - u) - y < 0.
\]

**Proof:** Straightforward calculations. We can now state the principal results concerning the dynamical system (42) – (44).

**Proposition 8:** The steady state of the above dynamical system (42) – (44) is locally asymptotically stable if \( s_c > i_1 \) and \( \beta_w \) is sufficiently small.

**Proof:** Due to \( y_m > 0, \rho u < 0 \), throughout, we have \(|J| < 0 \) and \( J_1, J_3 > 0 \) \((X_l^w < 0)\). A sufficiently small \( \beta_w \) will then also make \( J_2 > 0 \). Furthermore, the trace of \( J \) is given by

\[
(\tilde{\beta}_u y_u (x \lambda) - \tilde{\beta}_p y_u (y P) u - \tilde{x} \tilde{\nu}_m y_m (y P) + \tilde{\nu}_m y_m (x \lambda) - s_c \rho m) m
\]

where \( \rho m = y_m (1 - u) \). In the case \( s_c > i_1 \) we know that \( y_u \) will be positive while \( y_m, \rho m \) are always positive. Therefore, the components of trace \( J \) are negative except for its first term, which will not affect its sign of \( \beta_w \) is chosen sufficiently small. Consider finally

\[
b = (- \text{trace } J)(J_1 + J_2 + J_3) + \det J.
\]

39
It is easy to show that $b > 0$ holds when $\beta_w = 0$ is assumed. Thus, $b > 0$ for all $\beta_w$ sufficiently small. This in sum shows that the sufficient conditions for local asymptotic stability of the Routh–Hurwitz theorem all hold true here.

**Proposition 8′:** The steady state of the above dynamical system (42) – (44) is locally asymptotically stable if $s_c > i_1$ and $\beta_p$ is sufficiently large.

**Proof:** Since $|J|$ depends linearly on $\beta_p$, while trace $J$ and $J_1 + J_2 + J_3$ both depend linearly and positively on it, $(y_u > 0!)$ we can always choose a $\beta_p$ sufficiently large such that $J_2 > 0$, trace $< 0$ and $b > 0$ will be fulfilled.

The case $s_c > i_1$ can therefore always be stabilized by a sufficient degree of wage–inflexibility or price–flexibility. Of course, since only trace $J, J_2$ and $b$ can create stability problems in a fairly simple fashion there will exist a variety of further cases where local asymptotic stability will hold true.

If the parameter $h_2$ in the money demand function is chosen sufficiently small we not only know that inequalities (41) must hold, but get also (see (35) that

$$y_m \mapsto 1/h_1, \quad y_u \mapsto 0$$

as $h_2$ approaches zero. Thus trace $J$ must always be negative for $h_2$ sufficiently small. From this, we obtain in addition to the above propositions.

**Proposition 9:** The assertion of the propositions 8, 8′ will also hold true for $s_c < i_1$ ($y_u < 0$) if the parameter $h_2$ is chosen sufficiently small.

**Proof:** Similar to the proofs before, if one takes into account that the trace of $J$ is no longer an argument in the process of generating local asymptotic stability, which rather now only depends on $J_2$ and $b$.

We thus end up with the result, that the case of a small parameter $h_2$ always favors $\beta_p$ flexibility and $\beta_w$-inflexibility as carriers of local asymptotic stability, while the case of a large (infinite) parameter $h_2$ favored wage flexibility in one case (where price–flexibility was then destabilizing) and price–flexibility in another case (where wage–flexibility was bad for stability). This new asymmetry\(^\star\) between wage– and price–flexibility seems to be due to the Keynes–effect $y_m > 0$ (or $y_p < 0$) which was absent in the case $h_2 = \infty$ ($r \equiv r_0$).

We also note that assuming that both wages and prices will react to disequilibria in a sufficiently sluggish way will not always lead to local asymptotic stability, which will then depend on the relative size of $\beta_w$ and $\beta_p$ even for low values of these parameters.

**Proposition 10:** Assume $s_c > i_1$, i.e. $y_u > 0$. There is exactly one value $\infty \geq \beta_w^H > 0$ for the parameter $\beta_w$ which separates asymptotically stable steady states ($\beta_w < \beta_w^H$) from unstable ones ($\beta_w > \beta_w^H$). At $\beta_w^H$ a Hopf bifurcation occurs (if $\beta_w^H < \infty$).

\(^\star\)which now cannot change place with each other
**Proof:** Since both ±trace $J$ and $J_1 + J_2 + J_3$ are linear functions of $\beta_w$, the proposition is obvious should their intersection with the horizontal axis determine the value $\beta_w^H$ where stability gets lost for the first time. If, on the other hand, the quadratic function $b(\beta_w)$ is the determining expression for the $\beta_w^H$-value, then it can only be in the situation depicted in figure 12, where it crosses the positive part of the horizontal axis twice, since we already know $b(0) > 0$.

![Figure 12: Determination of the bifurcation parameter value $\beta_w^H$.](image)

When either ±trace $J$ or $J_1 + J_2 + J_3$ (which are linear functions of $\beta_w$) is strictly decreasing they must cut the horizontal axis between $\beta_w^{H_1}$ and $\beta_w^{H_2}$, i.e., local asymptotic stability cannot be reestablished after $\beta_w^{H_2}$ is crossed by means of a $b$ that eventually becomes positive again. If, however, both ±trace $J$ or $J_1 + J_2 + J_3$ are strictly increasing,\(^{24}\), we get for them the following functional form

$$-\text{trace } J = a_1 \beta_w + a_2, J_1 + J_2 + J_3 = b_1 \beta_w + b_2$$

where all coefficients are strictly positive. But then

$$(\text{±trace } J)(J_1 + J_2 + J_3) = a_1 b_1 \beta_w^2 + (a_1 b_2 + b_1 a_2) \beta_w + a_2 b_1$$

which gives a quadratic function that assumes its minimum at $\beta_w^{\text{min}} = \frac{-(a_1 b_2 + b_1 a_2)}{a_1 b_1} < 0$, i.e., the above depicted situation is then not possible. Therefore, no reswitching towards stability is possible once $\beta_w^{H_2}$ has been crossed. The proof of a Hopf-bifurcation is now a routine exercise.

\(^{24}\)we neglect here the border case of slope 0
Proposition 11: Assume $s_c < i_1$, i.e. $y_u < 0$ and that the dynamics are locally asymptotically stable for $\beta_w$ sufficiently small. There is exactly one value $\beta_w^H \geq \beta_w^H > 0$ for the parameter $\beta_w$ which separates asymptotically stable steady states ($\beta_w < \beta_w^H$) from unstable ones ($\beta_w > \beta_w^H$). At $\beta_w^H$ a Hopf-bifurcation occurs (if $\beta_w < \infty$).

Proof: Same as before.

Note here that similar results with respect to the parameter $\beta_p$ are not so easily obtained and if possible may work into the reverse direction. The case $s_c < i_1$ therefore demands further investigations for less restrictive choices of the parameter $h_2$.

Our above calculations have in particular shown that: (i) sign $|J|$ depends solely on the sign of the Keynes-effect $y_m$ (and $J_1$ also) where we always have $y_m > 0$, (ii) sign $J_3$ depends solely on the sign of the profitability-effect: $\rho_u$ ($< 0$ always). On the other hand, (iii) sign (trace $J$) depends on the signs of $y_m$ and $y_u$ as well as on the relative sizes of the parameters $\beta_w$, $\beta_p$ (in their interaction with the sign of $y_u$), (iv) sign $J_2$ depends on $y_m$ in interaction with the relative sizes of the parameters $\beta_w$, $\beta_p$.

The term $b = ( \text{trace } J)(J_1 + J_2 + J_3) + \det J$ is more difficult to judge and must be analyzed in the way we have done it in the proof of proposition 3. Should - trace $J$, $J_1$, $J_2$ and $J_3$ be all positive, it suffices to check $J_{12}J_{23}J_{31}$ and $J_{13}J_{21}J_{32}$ for positivity, since these are the only elements in $\det J$ which can then endanger the positivity of $b$.

\textsuperscript{25}The Rose-conflict between $\beta_w$ and $\beta_p$
7 Monetary and real factors in Keynesian cyclical growth dynamics: The 4D-case

In the previous sections we have shown that the qualitative features of the Keynes-Wicksell model discussed in the introduction are not completely overthrown when this supply side Keynesian model is modified to a demand side version of the determination of the output of firms. On the contrary, the cyclical growth dynamics of the Keynes-Wicksell case reappears in our Keynesian model, though it is embedded in a much more diversified structure of possible stability scenarios. This is perhaps not too surprising, since the differences between our Keynesian prototype and the Keynes-Wicksell model – though important from the viewpoint of proper theorizing – are few. The wage-price sector therefore still seems to play a dominant role in the determination of the dynamics of both classes of model.

In this section we introduce the differential equation for expectations to thus consider the full four-dimensional system. In subsection 7.1 we consider some extreme forms of the expectations mechanism. In subsection 7.2 we eliminate the real sector and consider the interplay of inflationary expectations and money which we find is also a cycle generating mechanism. In subsection 7.3 we isolate again the real sector and consider the introduction of a nonlinear investment equation and its role in the generation of cycles. Finally in subsection 7.4 we consider the full four-dimensional dynamics mostly by numerical simulations, but guided by our analysis of the various lower dimensional subcases.

7.1 The Role of Expectations

Here we briefly consider some extreme forms of the expectations formation mechanism.

At one extreme we would have Regressive Expectations ($\beta_{\pi_2} = 0, \beta_{\pi_1} < \infty$): This special case of our general expectation mechanism again does not modify significantly the results of the three dimensional case where expectations have been assumed to be always equal to the steady state rate of inflation.

At the other extreme we would have Adaptive Expectations ($\beta_{\pi_2} = 0, \beta_{\pi_1} < \infty$): In the case of adaptive expectations the four-dimensional dynamics becomes fully interdependent, since the evolution of $\pi$ now depends on $\omega, l$ and $m$ and that of $\omega, l$ and $m$ on $\pi$. The evolution of inflationary expectations $\pi$ is in this case determined by $\dot{\pi} = \beta_{\pi_1} (\dot{\pi} - \pi)$ where from section 2 we have $\dot{\pi} - \pi = \kappa [\beta_p (y(u, m, \pi)/y^P - 1) + \kappa_p \beta_{\omega}(y/(xl) - 1)]$. This gives for the dependence of the rate of change of $\pi$ on itself the expression

$$\dot{\pi}_\pi = \beta_{\pi_1} [\beta_p y^\pi /y^P + \kappa_p \beta_{\omega y^\pi}/(xl)].$$

This expression ($= J_{44}$ of the Jacobian of this extended dynamics) shows that the model of section 6 can again be made locally unstable via the addition of adaptive expectations by
choosing the parameter $\beta_{n_1}$ sufficiently high – if the Mundell effect $y_{e1}$ is normal ($y_{e1} > 0$), see again section 3 for alternative possibilities. As it is known from other models we thus get here too – under the assumption just made – that adaptive expectations create (at least locally) explosive behavior if they become sufficiently fast. 26 Yet, this situation is no longer as universal as in the Keynes-Wicksell case.

Another extreme case would be

Myopic perfect foresight ($\beta_{n_2} = 0, \beta_{n_1} = \infty$):

The fact that the trace of $J$ approaches $+\infty$ for $\beta_{n_1} \to \infty$ in the just considered case of adaptive expectations again indicates that the limit case $\beta_{n_1} = \infty$, i.e. $\pi = \hat{\pi}$, may be of a problematic nature. In this case, the two Phillips-type adjustment mechanisms (21), (22) of our general framework reduce to

$$\dot{\omega} = \beta_u (y/(x_1) - 1)$$
$$\kappa_p \dot{\omega} = \beta_p (y/(y_1) - 1)$$

where $y$ is determined as in section 2. This case therefore gives rise to two different and seemingly contradictory real wage dynamics if $\kappa_p > 0$ and $\beta_p < \infty$ holds true – unless labor market disequilibrium $V - 1$ and goods-market ‘disequilibrium’ are always proportional to each other with the proportionality factor $-\beta_p/(\beta_u \kappa_p)$. This implies that the utilization rates of the two factors of production are strictly inversely related to each other and therefore give rise to ‘perverse’ sort of Okun’s law. In the case $\kappa_p = 0$, by contrast, we have always full utilization of the capital stock and therefore another pronounced departure from the validity of Okun’s law. We conclude that the case of strict myopic perfect foresight is again problematic from an economic point of view as is known from a great deal of literature.

Finally consider the general case of forward and backward looking expectations ($\beta_{n_1} \in (0, 1), \beta_{n_2} \in (0, 1)$):

From a formal point of view this case represents the summation of the case of adaptive and regressive expectations and it thus inherits the stability and instability features of its two limit cases we have just discussed. These now combined expectation mechanisms can also be represented by the following formula:

$$\pi = \beta_{n_1} \alpha \hat{\pi} + (1 - \alpha) (\mu_0 - n) - \pi_0$$
$$\alpha = \frac{\beta_{n_1}}{\beta_{n_1} + \beta_{n_2}}, \beta_{n_1} + \beta_{n_2}.$$

In this form it says that a certain weighted average of the currently observed rate of inflation and of the future steady state rate is the measure according to which the expected medium run rate of inflation is changed in an adaptive fashion. Our discussion of the perfect foresight case suggests that $\alpha \neq 1$ should hold at all times so that medium run expectations of inflation are never governed by the short-run actual rate of inflation solely, but should always contain some non-myopic forward-looking component. By contrast, it is perfectly legitimate to set the adjustment speed parameter $\beta_{n_1}$ equal to $\infty$, which gives $\pi = \alpha \hat{\pi} + (1 - \alpha) (\mu_0 - n)$ as rule for inflationary expectations.

26Expectations on the medium run are here totally governed by knowledge about the short run – and will lead to instability if the lag between the short and the medium run becomes too short.
7.2 The purely monetary cycle

In this subsection we assume on the basis of the above discussion that the parameter values \( \beta_p, \beta_n, \beta_\pi \) are all positive and finite and thus exclude from consideration the one-sided cases we have just considered. We here also assume \( \mu_2 = \mu_0 = n \) for reasons of simplicity.

In order to derive the pure form of the monetary cycle we shall again make use of the following two sets of assumptions, (i) \( \beta_w = 0, \kappa_w = 1 \): The real wage is thereby made a constant of the model and in addition it is set equal to its steady state value, and (ii) \( \dot{K} = n (\dot{L} = n) \): The additional capacity effects of investment that are caused by profitability differentials (but not its trend component) is thereby suppressed on the supply side of the model (and only there). The labor intensity \( I = L/K \) thus is a constant in the following and it is set equal to its steady state value \( I^d \) in addition. Both sets of assumptions can be justified in the usual way by stating that the intent of the present investigation is confined to some idealised sort of medium run analysis, see for example Turnovsky (1977) in this regard. They here simply serve the reduction of the dimension of the above considered dynamics by two to two (in the variables \( m, \pi \)). The then resulting subdynamics read:\(^{27}\)

\[
\dot{m} = \mu_0 - n - \pi - \kappa \beta_p (y/y^p - 1), \\
\dot{\pi} = \beta_n \kappa \beta_p (y/y^p - 1) + \beta_\pi (\mu_0 - n - \pi),
\]

where the output-capital ratio \( y \) is given by:\(^{28}\)

\[
s_c(y(1-u_o) - \delta - t^n) = n + \alpha y(1-u_o) - \delta - \alpha y(1-u_o) - \delta - (r_0 + (h_y - m)/h_2) + \pi) + i_2(y/y^p - 1).
\]

The following investigation makes use of a nonlinear \( i_1 \)-component of the investment function \( i(\cdot) \) of the following type:

\[
i_1(\rho - r + \pi) = \tan(\alpha * Pi * (\rho - r + \pi)), / (\alpha * Pi), \quad \rho = y(1-u_o) \quad \alpha > 0.
\]

With respect to this investment function one gets as in section 3 for the partial derivatives of the function \( y(m, \pi) \) implicitly defined by the above goods-market equilibrium condition:\(^{29}\)

\[
y_m = \frac{n + i_1'(\cdot)/h_2}{(s_c - i_1'(\cdot)(1-u_o) + i_1'(\cdot)h_1/h_2 - i_2/y^p)}, \\
y_\pi = \frac{i_1'(\cdot)}{(s_c - i_1'(\cdot)(1-u_o) + i_1'(\cdot)h_1/h_2 - i_2/y^p)}\]

These partial derivatives are well-defined and non-negative if the following conditions hold:

\[
s_c(1-u_o) - i_2/y^p > 0, \quad h_1/h_2 - (1-u_o) > 0,
\]

since the denominator in the above fractions is then always positive (and larger than \( s_c(1-u_o) - i_2/y^p \)). In this case we always have a normal Keynes-effect as well as a normal Mundell-effect associated with the considered dynamics.

---

\(^{27}\)Note that the output-capital ratio and the rate of profit is not a constant in the present context.

\(^{28}\)\( u_o = \omega_o / z \)

\(^{29}\)Note that this function is linear if this holds for the investment term \( i_1(\cdot) \).
The steady state of the dynamics is the same as before and its Jacobian at the steady state is given by:

\[ J = \begin{pmatrix}
-\kappa \beta P y m / y^P m & -(1 + \kappa \beta P y m / y^P m) \\
\beta_{n_2} \kappa \beta P y m / y^P & -\beta_{n_2} + \beta_{n_1} \kappa \beta P y m / y^P
\end{pmatrix} \]

The sign of the determinant of this Jacobian is easily shown to equal the sign of \( y_m \) and is thus positive, while the trace of \( J \) is given by

\[ \kappa \beta P (\beta_{n_1} y m - y m^2) / y^P - \beta_{n_2}. \]

For \( \beta_{n_1} = 0 \) we therefore get a positive determinant and a negative trace of the matrix \( J \) and thus local asymptotic stability of the steady state. It is easily shown furthermore that the dynamics undergo a Hopf bifurcation when the parameter \( \beta_{n_1} \) is chosen sufficiently large – due to the dominance of the Mundell effect \( y m \) that then comes about.

The following construction of conditions that imply the validity of the Poincaré-Bendixson theorem gives, however, rise to a situation that is much more general than that of a Hopf (limit) cycle (or that of a Hopf closed orbit structure) at some intermediate value of the parameter \( \beta_{n_1} \). To this end we have to calculate the slopes of the isoclines for the \((\pi, m)\)-phase diagram of the above dynamics first. They are given by:

\[
\begin{align*}
\dot{m} &= 0: \quad m'(\pi) = -\frac{y m + \frac{1}{\kappa \beta P y^P}}{y m} < 0 \\
\dot{\pi} &= 0: \quad m'(\pi) = -\frac{y m - \frac{1}{\beta_{n_1} \kappa \beta P y^P}}{y m} \geq 0
\end{align*}
\]

These isoclines can give rise to the phase diagram in figure 13 for this purely monetary cycle model.
Figure 13:
The purely monetary cycle.

The invariant box shown in this diagram may be obtained by choosing the parameter $\beta_{\pi_0}$ appropriately large such that the $\hat{\pi}$-isocline cuts the horizontal axis just once (to the left of the steady state value $\pi_o = 0$). If this is given, the shown rectangle can be constructed by following the sequence of points $A, B, C, D$. Note here that the conditions on the parameter $\beta_{\pi_0}$ may be such that the slope of the $\hat{\pi}$-isocline is positive throughout, in which case there is a stable steady state and thus no compelling reason for the existence of a limit cycle in the above diagram. In the opposite case (where the above depicted situation holds) the existence of at least one limit-cycle – and the proposition that all trajectories in the above domain are attracted by one such cycle (or identical to it) – are an immediate consequence of the Poincaré-Bendixson theorem. The assumptions made indicate, however, that the scope for the application of this theorem in the present context may be small.

Figure 14 shows a numerical investigation of this limit growth cycle which is based on the parameter restrictions we have discussed above.\(^{30}\)

\(^{30}\)The parameter values of this simulation which is performed with the step length $h = .01$ by means of the simple Euler method are:

\[
\begin{align*}
\sigma_c &= .8, \delta = .1, t^\alpha = .35, n = \mu_0 = \mu_2 = .05, h_1 = .1, h_2 = .1, y = 1, x = 2, l^d = .5 \\
\mu_1 &= 0 = \theta, \beta_w = 0, \beta_p = 1, \kappa_w = 1, \kappa_p = .5, \beta_{\zeta_1} = 1.1, \beta_{\zeta_2} = .3, i_1 = 1, i_2 = .3, \alpha = 20.
\end{align*}
\]
These collection of plots shows a monetary (limit) cycle and it also shows the exploited nonlinearity of the investment function in the generation of this cycle. The lower two pictures add some time series plots to this numerical example of the $m, \pi$-dynamics. Note here that the amplitude of the fluctuations of the rate of inflation is fairly large in this purely monetary cycle.

7.3 Investment Nonlinearity and the Real Cycle

In this subsection we assume as in the previous one that the parameter values $\beta_p, \beta_\pi$ are positive and finite and also that $\mu_2 = 0$ holds in order to remove now monetary influences from the real part of the cycle as in section 6 ($\mu_0 = n$ again for reasons of simplicity).

The other assumptions of section 6 that were used for this purpose and are retained here were, (i) $h_2 = \infty$, i.e., $r = r_0$, and, (ii) $\beta_\pi = \infty$, i.e., $\pi = \mu_0 - n = 0$. These two assumptions
here simply serve to reduce the dimension of the full dynamics by two to two (now in the variables \( u = \omega/x, l \)), thereby again allowing a preliminary investigation here of the real part of the model (as an isolated substructure of the full dynamics).\(^{31}\)

The resulting autonomous subdynamics in the variables \( u, l \) read:

\[
\begin{align*}
\dot{u} & = \kappa[(1 - \kappa_p)\beta_p(y(u)/(zl) - 1) + (\kappa_w - 1)\beta_p(y(u)/y^p - 1)], \\
\dot{l} & = n - s_c(y(u)(1 - u) - \delta - t^n),
\end{align*}
\]

with the output-capital ratio \( y(u) \) implicitly given by

\[
s_c(y(u)(1 - u) - \delta - t^n) = n + i_1(y(u)(1 - u) - \delta - r_0) + i_2(y(u)/y^p - 1).
\]

This model has been extensively studied in section 6 with respect to a linear shape of this implicitly defined \( y(u) \)-relationship (and linear or nonlinear market adjustment functions). It will now be briefly investigated by assuming as in the preceding subsection the following nonlinear shape for the \( i_1 \)-component of the employed investment function \( i(\cdot) \) in the place of the nonlinearities we employed in section 6:

\[
i_1(\rho - r_0) = \text{atan}(\alpha \star P_i(\rho - r_0))/(\alpha \star P_i), \quad \rho = y(u)(1 - u) - \delta, \quad \alpha > 0.
\]

This function of \( \rho \) is strictly increasing and zero at \( \rho = r_0 \) and it approaches two bounds \( c, -c, \ c > 0 \) as \( \rho \to \pm \infty.\)\(^{32}\) Furthermore, there exist exactly two values \( \rho_1, \rho_0 \) of \( \rho \) where the slope of this function is equal to \( s_c \) — if its parameter \( \alpha \) is chosen so large that \( s_c \) is smaller than its slope at \( \rho = r_0. \) The figure 15 summarizes this situation.

\(^{31}\)Of course, the consequences of less extreme assumptions on interest elasticity and inflationary expectations have to be investigated carefully to see in how far the results obtained for the partial model survive the integration with the monetary part of the full dynamics.

\(^{32}\)See figure 14 for a numerical plot of this function for \( \alpha = 2. \)
Figure 15:
The nonlinear component of the investment function.$^{33}$

With respect to such a reformulated investment function one gets (as in section 3) for the derivative of the function $y(u)$ defined by the above goods-market equilibrium condition (47):$^{34}$

$$y'(u) = \frac{(s_c - i_1(\cdot))y(u)}{(s_c - i_1(\cdot))(1 - u) - i_2/y^P} = \frac{N}{D}.$$  

(48)

This derivative – and the function $y(u)$ – are locally well-defined around the steady state $u_0$ if $D = s_c(1 - u_0) - i_2/y^P < 0$ holds true, which is assumed in the following.$^{35}$ Closer inspection of the denominator $D$ of this derivative furthermore shows that it – and the function $y(u)$ – is well-defined on the interval $(u_c, 1)$, where the value of $u_c$ is given by the first point to the left of $u_0$ where the denominator $D$ vanishes (this will happen at a value $u_c > 0$ if $s_c > i_2/y^P + i_1'(y(0) - r_0)$ is assumed in addition.) We shall work in the following with the opposite assumption, i.e., we assume that $u_c < 0$ holds. Let us provisionally here also assume for the following that the function $y(u)$ fulfills the conditions $y(1) < 0, y(0) > y^P, y(u) > 0$ for $u < u_0$.$^{36}$ Due to equation (48) it must therefore have the shape shown in figure 16.

$^{33}$Note that we have $y = y^p$ when the above investment curve intersects the depicted savings line.

$^{34}$Note that this function is linear if this holds true for the investment term $i_1(\cdot)$.

$^{35}$This assumption is the opposite of an assumption we made in the subsection on the purely monetary cycle.

$^{36}$The first condition is implied by $s_c(\delta + i^n) + n \geq i(\delta - r_o - \delta) + i_2$. 

50
The values \( \bar{u}, \underline{u} \) in this figure can be determined in the following way: It is easily obtained from the above equation (47) – by inserting \( \rho = y(1 - u) - \delta \) into it – that the derivative of the thereby defined function \( \rho(u) \) is given by

\[
\rho'(u) = \frac{i_2 y/y^p}{(s_c - i_1(\cdot))(1 - u) - i_2/y^p}.
\]

This derivative is strictly negative on the whole interval \((0, u_c), u_o < u_c < 1\) where the values of the function \( y(u) \) are positive (see the above figure). It follows that there exist uniquely determined values \( u, \bar{u} \) in \((0, u_c)\) such that \( \rho(u) = \rho, \rho(\bar{u}) = \bar{\rho} \), since \( y(u) \) passes once again through \( y^p \) to the left and to the right of \( u_o \).

Let us now calculate the isocline \( \dot{u} = 0 \). As in section 5 it is easily shown to be of the form:

\[
l = \frac{1/x}{(1 - q)/y(u) + q/y^p},
\]

where as previously \( q = (1 - \kappa_u)\beta_p/((1 - \kappa_p)\beta_w) \). It follows that this expression is always well defined in the above situation \((u \in (0, u^c))\) when \( q < 1 \) is assumed. The derivative of this isocline reads

\[
l'(u) = \frac{(1 - q)y'(u)}{[(1 - q)/y(u) + q/y^p]^2xy(u)^2},
\]

\(^{37}\) and \( y = (\rho + \delta)/(1 - n) \)
i.e. the slope of this isocline has the same sign as that of the \( y(u) \)-curve shown above. It furthermore cuts the horizontal axis at the same value of \( u \) as the \( y(u) \) curve, though it will still be finite in value when the former curve is infinite (at \( u = u_c < 0 \)).

The assumed situation thus gives rise to the phase diagram in figure 17 for the dynamics (45) – (46):

![Phase Diagram](image)

**Figure 17:**

*The phase diagram of a purely real cycle.*

This phase diagram suggests that the conditions of the Poincaré-Bendixson theorem are fulfilled in the assumed situation, since (i) no trajectory can leave the positive orthant in finite time, and (ii) a trajectory that connects \( A \) with \( A' \) can always be found. The set enclosed by ABCD is therefore an invariant set of the considered dynamics, that is – due to the instability of its unique steady state – each limit set of each trajectory which starts in the domain must be a closed curve.

The figures 18 provide a simulation study of this real (partial) limit cycle. To the top right it shows the nonlinear \( y(u) \)-function that is generated by the nonlinearity in the investment function for those \( u \)-values that are in fact reached through the depicted trajectory. Remarkable in these figures is also the fact that the rate of capacity utilization is not closely correlated with the rate of employment of the labor force. The rate of profit, by contrast, moves strictly inversely to the share of wages (though the function \( y(u) \) is not monotonic).
Figure 18:
A simulation of the purely real cycle.\textsuperscript{38}

7.4 The Real and the Monetary Cycle

We have seen that nonlinearities in the investment function are difficult to treat even in the two partial and autonomous two-dimensional cases we have considered above.

For the integrative 4D-case it is at this stage of the investigation only possible to prove - as in section 6 - that the determinant of the Jacobian of this dynamics is always strictly positive\textsuperscript{39} and that local asymptotic stability will come about - by means of continuity arguments - if the the related situation of the preceding 3D-case is locally asymptotically stable and the

\textsuperscript{38}The parameter values of this simulation are the following:

\[ s_c = .8, \delta = .1, \tau^n = .35, n = \mu_0 = .05, h_1 = .1, h_2 = 10000, y = 1, x = 2, \]
\[ \mu_1 = 0 = \theta = \mu_2, \beta_w = 2, \beta_p = 1, \kappa_w = .5, \kappa_p = .5, \beta_{\pi} = 0 = \beta_{\pi 2}, i_1 = 2, i_2 = 2, \alpha = 10. \]

\textsuperscript{39}that of the 5D-case is always strictly negative!
parameter $\beta_{\pi}$ is sufficiently small. On this basis the Hopf-bifurcation will again be applicable with respect to the parameter $\beta_{\pi}$. We do not go into the details of such an application here, but only note that the assumptions of section 6 will give rise to a positive Mundell-effect in the present situation which is the basis for the Hopf-bifurcation result just stated.

We conclude this paper with a numerical presentation of the 4D dynamics. The figures 19, 20 provide a first brief impression how an explosive situation $\beta_{\pi} = .6$ in the "linear" case may be tamed by the only economic behavioural nonlinearity (in the investment function) we have considered in the preceding section. The plots below show to the right the real cycle and to the left the time series of the four state variables of the "linear" version of the model.

![Graphs showing the real cycle and time series of state variables](image)

**Figure 19:**

A simulation of the joint monetary and the real cycle in the "linear" case.

The parameter values of this "linear" case are the following:

\[
\begin{align*}
sc &= .8, \delta = .1, t^m = .35, n = \mu_0 = \mu_2 = .05, h_1 = .1, h_2 = 10000, y = 1, x = 2, \\
\mu_1 &= 0 = \theta, \beta_w = 1.3, \beta_p = .5, \kappa_w = .2, \kappa_p = .1, \beta_{\pi_1} = .6, \beta_{\pi_2} = 0, i_1 = 1, i_2 = .8.
\end{align*}
\]

Note that we here still stick to the assumption of a fairly inflexible nominal rate of interest ($h_2$ large). Assuming parameter $h_2$ considerably below this value means in the present situation that asymptotic stability of the steady state is established, so that the model then exhibits fluctuations with ever-decreasing amplitude.
In the following nonlinear case we use the atan-function of the preceding subsections with the value 2 for the parameter $\alpha$, and keep all other parameter values unchanged. The plots below again show to the right the real cycle and to the left the time series of the four state variables of the model.

![Graphs showing the nonlinear relationship between rate of employment and wage share, and time series plots of various economic variables.](image)

**Figure 20:**

*A simulation of the joint monetary and the real cycle in the nonlinear case.*

The above plots show that the explosive cycle in the preceding figure can indeed be 'tamed' such that a 'limit cycle' at least for the real subsector is obtained.

## 8 Conclusions

We have introduced in this paper an integrated or complete Keynesian model of monetary growth with both sluggish wage and price adjustments. Temporary IS-LM goods and money market equilibrium determines the rate of utilization of both labor and capital and wage and price adjustment is then formulated by means of two instead of only one expectations augmented Phillips-curve in the usual demand-pull fashion. Inflationary expectations were assumed as being of a both backward as well as forward looking type.

It was shown that the real sector of this economy allows for a variety of situations, in
particular of a growth cycle type as in Goodwin (1967) or more generally as in Rose (1967), the latter a limit cycle result when appropriate nonlinearities are added to the wage or price adjustment equations in the presence of local asymptotic instability caused by the price or wage adjustment, respectively, and when an appropriate nonlinearity in the investment demand function is employed. Furthermore, also the monetary sector can be shown to imply persistent limit cycle behavior, if the Cagan type instability it contains is limited again by means of this nonlinearity in the investment function.

Combining real and monetary factors can be done in scenarios of increasing complexity leading from 3D to 4D and finally to 5D dynamical systems. Common to all these systems is that the determinant of their Jacobian has always the sign that is demanded by the Routh-Hurwitz conditions of local asymptotic stability, so that stability can only get lost by way of a Hopf-bifurcation even in these higher dimensional cases. On this basis we have provided in this paper a variety of stability assertions for the 3D case and have briefly indicated how these assertion can be carried ove to the 4D situation together with a first numerical presentation of this latter situation.

Yet, up to that, 4D and even more 5D situations must here be left for future investigations which should also be built on a more detailed analysis of the basic monetary growth situation here considered, that is the monetary growth cycle in the variables $\omega$, the real wage, $I$, the relative factor intensity, and $m$, real balances per unit of capital.

In sum, one may nevertheless already state, that this complete and even in its basic form nonlinear Keynesian monetary growth prototype model exhibits a high potential for cyclical adjustment patterns which will often give rise to super- or subcritical Hopf-bifurcations and which may demand further nonlinearities in order to get a viable dynamics in other cases, where only explosive cycles are at first generated.
Appendix 1: Summary of Notation

A. Statically or dynamically endogenous variables:

\[ Y > 0 \] Output (\( Y \neq Y_P = \) potential output)
\[ Y_{c,d}^{*,*} \] \( * \): perceived disposable income (index \( c \): of asset-holders)
\[ L^d > 0 \] Employment
\[ C > 0 \] Consumption
\[ I \] Investment
\[ r > 0 \] Nominal rate of interest (price of bonds \( p_B = 1 \))
\[ p_E > 0 \] Price of Equities
\[ S \] Total savings
\[ S_p > 0 \] Private savings
\[ S_g \] Government savings
\[ T > 0 \] Real taxes
\[ G > 0 \] Government expenditure
\[ \rho \] Rate of profit (before taxes, \( \rho^n \) after taxes)
\[ \zeta \] Real rate of interest (before taxes, \( \zeta^n \) after taxes)
\[ V = \frac{L^d}{L} \] Rate of employment (\( \bar{V} \) the first NAIRU concept)
\[ U = \frac{Y}{Y^p} \] Rate of capacity utilization (\( \bar{U} \) the second NAIRU concept)
\[ K > 0 \] Capital stock
\[ w > 0 \] Nominal wages
\[ p > 0 \] Price level
\[ \pi \] Expected rate of inflation (long-run)
\[ M > 0 \] Money supply (index \( d \): demand, growth rate \( \mu \))
\[ L > 0 \] Normal labor supply
\[ B > 0 \] Bonds (index \( d \): demand)
\[ E > 0 \] Equities (index \( d \): demand)
\[ W > 0 \] Real Wealth
\[ \omega > 0 \] Real wage (\( u = \omega/x \) the wage share)

B. Exogenous variables and parameters

\[ \delta > 0 \] Depreciation plus inventory rate
\[ i_{1,2} > 0 \] Investment parameters
\[ h_{1,2} > 0 \] Money demand parameters
\[ n > 0 \] Natural growth rate
\[ \mu_0 > 0 \] Steady growth rate of money supply
\[ \mu_1 > -1 \] Adjustment parameter in view of expected inflation
\[ \beta_w > 0 \] Wage adjustment parameter
\[ \beta_p > 0 \] Price adjustment parameter
\[ \beta_{\pi,1,2} > 0 \] Inflationary expectations adjustment parameters
\[ \kappa_{w,p} \in [0,1], \kappa_{w,p} \neq 1 \] Weights of short- and long-run inflation (\( \kappa = (1 - \kappa_{w,p})^{-1} \))
\( y^p > 0 \)  
Potential output–capital ratio (\( \neq y \), the actual ratio)

\( x > 0 \)  
Output–labor ratio

\( \tau \in (0,1) \)  
Tax rate = const. (or \( t^n = (T - rB/p)/K = \text{const.} \))

\( s_c \in [0,1] \)  
Savings–ratio (out of profits and interest)

\( s_w \in [0,1] \)  
Savings–ratio (out of wages, = 0 here)

\( \theta, \mu_2 \)  
Fiscal policy parameters

C. Mathematical notation

\( \dot{x} \)  
Time derivative of a variable \( x \)

\( \ddot{x} \)  
Growth rate of \( x \)

\( l', l_w \)  
Total and partial derivatives

\( y_w = y'(l)l_w \)  
Composite derivatives

\( r_o, \text{etc.} \)  
Steady state values (\( \bar{r} \) parameter which may differ from \( r_o \))

\( y = Y/K, \text{etc.} \)  
Real variables in intensive form

\( m = M/(pK), \text{etc.} \)  
Nominal variables in intensive form
References/Bibliography


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