Learning Dynamics in a Nonlinear Stochastic Model of Exchange Rates

Carl Chiarella
Alexander Khomin

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by

Carl Chiarella and Alexander Khomin

School of Finance and Economics
University of Technology, Sydney
PO Box 123
Broadway 2007
NSW
Australia

Tel. +61 2 330 3626
Fax +61 2 330 3636
e-mail C.Chiarella@uts.edu.au
A.Khomin@uts.edu.au

Abstract

This paper considers a version of the Dornbusch model of exchange rate dynamics which allows a nonlinear domestic demand for foreign assets function and imperfect substitutability between domestic and foreign interest bearing assets. Expectations of exchange rate changes are modelled as adaptive with perfect foresight being obtained as a limiting case. For sufficiently rapid speed of adjustment of expectations the model is able to generate cyclical behaviour of the exchange rate and expectations of its change. In the perfect foresight limit the cycles become relaxation cycles. To this underlying model of the fundamentals a white noise "news" process is added. Agents are assumed to attempt to learn about the system dynamics and the link between such learning and exchange rate volatility is studied. Two learning scenarios are considered. In the first scenario economic agents are regarded as a
uniformly well-informed group of sophisticated traders. In the second scenario a group of "naive" traders coexists with the sophisticated traders. We find that both learning scenarios lead to increased volatility. However this effect increases in proportion to the weight of the "naive" traders.
1. Introduction

There continues to be some interest in nonlinear dynamic specifications of exchange rate movements. This interest is driven by both empirical and theoretical considerations.

The empirical motivation arises from a number of empirical studies which detect some form of nonlinear dependence in nominal exchange rate data; see eg. Hsieh (1989), Chiarella, Peat and Stevenson (1994) and Guillaume (1994). Another set of studies by Taylor (1994) claims that technical trading rules are profitable in foreign exchange markets. Whilst such studies do not necessarily imply a nonlinear explanation they do at least cast doubt on the traditional random walk paradigm.

The theoretical motivation arises from the characteristic saddle-point instability behaviour of descriptive models of exchange rate dynamics (usually based on the asset markets view) which incorporate the perfect foresight assumption. This modelling approach requires the imposition of arbitrary jumps in initial values in order that the exchange rate dynamics operate on a stable manifold. The rationale for this so-called "jump-variables" technique has never been clear and has remained an unsatisfactory feature of this approach. It has been accepted by authors who believe that the exchange rate must inexorably be driven to the rational expectations (or in a deterministic setting - perfect foresight) equilibrium. At this point it is worth citing the empirical study of Papell (1992) who in a study of seven exchange rates is unable to choose empirically between rational and adaptive expectations.

Chiarella (1990) develops a model of exchange rate dynamics of the Dornbusch (1976) variety which allows a nonlinear domestic demand for foreign assets function and imperfect substitutability between domestic and foreign interest bearing assets. Expectations of exchange rate changes are modelled as adaptive with perfect foresight being obtained as a limiting case. The resulting exchange rate dynamics are either stable to, or exhibit a limit cycle about, the equilibrium point depending upon the speed of adjustment of expectations of exchange rate changes. In the perfect foresight limit the jump-variable situation of the linear modelling framework turns out to be a relaxation cycle on which jumps in expectations and sudden changes of direction of exchange rate movements occur in an endogenous fashion.

The nonlinear model just described provides an idealised explanation of the source of nonlinear fluctuations in exchange rate time series. However to move closer to a model of observed exchange rates a number of features need to be added. Firstly the modelling of the foreign exchange market could be elaborated upon in a variety of ways indicated in the next section. Secondly, noise (capturing for example the incessant arrival of "news" onto the foreign exchange market) needs to be incorporated into the analysis since the "true" model of exchange rate dynamics is surely both nonlinear and stochastic. Thirdly, the interaction with the foreign exchange rate dynamics of attempts by economic agents to "learn" about the system dynamics needs to be incorporated in some way as casual empiricism would indicate.
that such learning (or adaptive) behaviour is an important feature of foreign exchange market dynamics.

In this paper we attempt to incorporate the interaction effects of economic agents trying to learn about the foreign exchange rate dynamics. In so doing we also incorporate some (additive) noise effects. The main issue we seek to address is whether the interaction effects of learning tend to increase or decrease the exchange rate fluctuations arising from the inherent nonlinear dynamics in the model. Our finding is that whilst the underlying nonlinear trend is barely affected, the effect of the noise on the system dynamics is amplified. This is so if the economic agents are regarded as a uniformly informed group of "sophisticated" traders. However if we introduce a group of "naive" traders alongside the "sophisticated" traders then the nonlinear fluctuations could increase in amplitude depending on the relative size of these two groups.

These results are in keeping with the analysis of Tabellini (1988) who, using a stochastic linear rational expectations model, found that learning magnifies the reaction of exchange rates to noise in the market fundamentals.

2. A Nonlinear Model of Exchange Rate Dynamics

Our basic economic setting is that of Dornbusch (1976), however in order to introduce the appropriate nonlinearity into the model we incorporate the portfolio approach to the modelling of the foreign exchange market as developed by Kouri (1983). Our exposition of the model follows the notation and approach of Gray and Turnovsky (1979) which shows very clearly the saddle-point structure of the Dornbusch model under perfect myopic foresight.

Let

\[ R^* = \text{foreign (nominal) rate of interest (assumed exogenous)}, \]
\[ R = \text{domestic (nominal) rate of interest}, \]
\[ E = \text{logarithm of the current exchange rate (units of domestic currency per unit of foreign currency)}, \]
\[ X = \text{expected rate of exchange depreciation}, \]
\[ M = \text{logarithm of the domestic nominal money supply (assumed exogenous)}, \]
\[ P = \text{logarithm of the domestic price level}, \]
\[ Y = \text{logarithm of domestic real output (assumed to be fixed)}. \]

Consider first of all the money market where we make the standard assumption that the
demand for real balances is an increasing function of output and a decreasing function of the nominal interest rate, i.e.

\[ M - P = \alpha_1 Y + \alpha_2 R, \quad (\alpha_1 > 0, \alpha_2 < 0). \] (1)

Next consider the goods market where demand, \( D \), is assumed to be an increasing function of output \( Y \), relative price \( (E - P) \), and a decreasing function of domestic nominal interest rate \( R \), i.e.

\[ D = \beta_0 + \beta_1 Y + \beta_2 R + \beta_3 (E - P), \quad (0 < \beta_1 < 1, \beta_2 < 0, \beta_3 > 0). \] (2)

We assume sluggish price adjustment in the goods market according to which the domestic price adjusts to excess demand, so that

\[ p^* = \rho [\beta_0 + (\beta_1 - 1)Y + \beta_2 R + \beta_3 (E - P)], \quad (\rho > 0). \] (3)

The essential feature of our nonlinear model is the manner in which the foreign exchange market is modelled. We adopt the portfolio approach of Kouri (1983), in which domestic residents hold some fraction of domestic marketable wealth in foreign assets. This fraction depends on the relative expected return and risk characteristics of domestic and foreign assets, which in the deterministic framework we are adopting is measured by the real interest
differential $R^* + X - R$. Thus taking the prices of foreign assets as exogenous in foreign currency and the stock of domestic marketable wealth as exogenous in domestic currency then the stock demand for foreign assets may be written.

$$\tilde{F}^d = \tilde{f}(R^* + X - R) \frac{V}{s}, \quad (4)$$

where $V$ stands for the value of domestic marketable wealth in domestic currency, $s$ is the exchange rate ($\ln s = E$) and $\tilde{f}$ is the fraction of $V$ domestic residents desire to hold in foreign assets. We assume that this fraction is an increasing function of the real interest differential $R^* + X - R$ and is bounded above and below as $R^* + X - R$ tends to large positive or negative values, i.e.

$$\tilde{f}'(R^* + X - R) > 0, \quad (5a)$$

and

$$\tilde{f}_1 < \tilde{f} < \tilde{f}_u. \quad (5b)$$

The upper and lower limits $\tilde{f}_u$ and $\tilde{f}_1$ arise from the assumption that domestic residents will hold domestic marketable wealth neither entirely in foreign assets nor entirely in domestic assets as the interest differential $R^* + X - R$ assumes large values. Rather some fixed proportion of wealth will be held in domestic assets and some in foreign assets, perhaps to satisfy transactions demand. It is also possible that the central bank can change the relationship between $\tilde{f}_u$ and $\tilde{f}_1$ by intervention in the foreign exchange market.

Taking logarithms of (4), the log of domestic demand for foreign assets can be written

$$\log \tilde{F}^d = \tilde{f}(R^* + X - R) - E + \log V, \quad (6)$$

where $\tilde{f} = \log \tilde{f}$, $\tilde{f}' > 0$ and $\tilde{f}_1 < \tilde{f} < \tilde{f}_u$. The general shape of $\tilde{f}$, displaying the essential features of positive slope and upper and lower bounds is shown in figure 1. This demand function, unlike the one implicit in the Dornbusch class of models, allows for imperfect substitutability between foreign and domestic interest bearing assets.

The other element of the foreign exchange market which we include in the model that we analyse below is the notion that there is gradual rather than instantaneous adjustment of
portfolios of foreign and domestic interest bearing assets. Empirical evidence of significant adjustment lags in portfolios has been presented by Kearney and MacDonald (1985) in their estimation of a structural model of the sterling-dollar exchange rate. Theoretical arguments favouring gradual portfolio adjustment include; transactions costs, the view that in portfolio balance models money serves as a buffer asset or signalling device, and; the view that in an aggregate model aggregate quantities may adjust with lags due to difference in timing by individual agents who may be maintaining continuous portfolio equilibrium (see e.g. Davis and Lewis (1977)). The effect of gradual portfolio adjustment on traditional models of exchange rate dynamics is analysed by Karacaoglu and Ursprung (1988).

In order to capture the gradual portfolio adjustment we assume that the exchange rate adjusts to excess demand for foreign exchange according to

$$\tau_e x = f(R^* + X - R) - E + \log V - S, \quad (7)$$

where $\tau_e > 0$ and $S$ is the fixed supply of foreign exchange. The parameter $\tau_e$ is the time lag with which the exchange rate adjusts to excess demand. In practice we would expect $\tau_e$ to be small but not zero. It is more traditional to formulate dynamic exchange rate models in terms of speed of adjustment parameters, $\gamma_e = 1/\tau_e$. For the moment however we prefer the time lag interpretation since in our analysis of the perfect foresight limit we need to concentrate on the behaviour of the dynamics as certain time lags tend to zero. It should be noted that we are concentrating on the special case, also considered by Kouri, in which foreign residents do not hold domestic assets and there is a fixed supply of foreign exchange. The assumption of fixed supply of foreign exchange is not as restrictive as it may appear. It is equally possible to model the demand of foreign residents for domestic interest bearing assets as a function of the interest differential. This function will also have upper and lower bounds for similar reasons that the function $f$ does. When the resulting supply function is introduced into equation (7) the $f(\bullet) - S$ term is replaced by an aggregate excess demand function still having the same shape as $f$. The condition $R^* + X - R = 0$ of the standard Dornbusch model is obtained when the demand function $f$ tends to the perfect substitutability limit discussed earlier (where $f$ becomes a step function).

The final element of our model is the mechanism for the formation of the expectations variable $X$. We shall assume that expectations are formed according to the adaptive expectations scheme

$$\tau_X \dot{X} = E - X, \quad (8)$$

where $\tau_X > 0$ is the time lag in the formation of expectations. In the limiting case $\tau_X = 0$ the conventional perfect foresight mechanism

$$X = E, \quad (9)$$
It is generally argued that in a world of rational economic agents only the perfect foresight mechanism (9) is relevant. However a major part of the contribution in Chiarella (1990) is to show that when proper account is taken of the nonlinear nature of the demand function discussed above, the exchange rate dynamics are qualitatively the same whether $\tau_X = 0$ or $\tau_X$ is positive and small.

We now turn to the analysis of the exchange rate model in which the exchange rate adjusts to excess demand according to (7) and exchange rate expectations are formed according to (8).

The dynamics of the exchange rate model can be expressed by the differential equations

\[
\begin{align*}
E &= \gamma_e \left[ f(R^* + X - R) - E + \log V - S \right], \\
X &= \gamma_X [E - X], \\
P &= \rho \left[ \beta_0 + (\beta_1 - 1)Y + \beta_2 R + \beta_3 (E - P) \right].
\end{align*}
\]

with $R$ being determined as a factor of $P$ from the money demand equation (1), i.e.

\[ R = (M - \alpha_1 Y)/\alpha_2 P/\alpha_2. \]  

Note that we now write the differential equations for $E$ and $X$ in terms of the speeds of adjustment $\gamma_e, \gamma_X$.

We define the deviation variables

\[
\begin{align*}
e &= E - E, \\
x &= X - X, \\
p &= P - P \text{ and } r &= R - R,
\end{align*}
\]

where the - indicates equilibrium values. In terms of these variables the differential system (12) may be written

\[
\begin{align*}
\dot{e} &= \gamma_e \left[ \phi(x + p/\alpha_2) - e \right], \\
\dot{x} &= \gamma_X [\gamma_X \left[ \phi(x + p/\alpha_2) - e \right] - x], \\
\dot{p} &= \rho \left[ - (\beta_2 + \alpha_2 \beta_3) p/\alpha_2 + \beta_3 e \right],
\end{align*}
\]

where the function $\phi$ in defined according to

\[ \phi(z) = f(n + z) - f(n), \]
and
\[ n = R \cdot \left( \frac{M - \alpha_1 Y}{\alpha_2} \right) + \frac{p}{\alpha_2} = R \cdot \bar{R}. \]  

(13e)

In Chiarella (1990) we have analysed the three dimensional nonlinear system (13) and shown how it may generate limit cycle behaviour for the expectations time lag \( \tau_x \) sufficiently small. We also showed in that paper how the use of the jump-variable technique may be avoided, since the limit cycles are stable. Furthermore when the expectations time lag \( \tau_x \) tends to zero to yield the perfect foresight limit the limit cycle was shown to tend to a relaxation cycle.

3. **Naive Learning**

If economic agents in foreign exchange markets were witnessing the underlying cyclical trend implied by the model of the previous section they would come to expect a change in the trend of \( e \) for \(|e - x|\) large. They might in that case be likely to reduce the absolute value of the change in \( x \) the larger is \(|e - x|\). Such behaviour is simply captured by replacing the expectations mechanism in (13b) with

\[ x = \gamma_x (\text{ref}(e) - x). \]  

(13b’)

where

\[ \text{ref}(e) = \begin{cases} 
\alpha_e e + (1 - \alpha_e) d_e & \text{if } e > d_e, \\
e & \text{if } |e| \leq d_e, \\
\alpha_e e - (1 - \alpha_e) d_e & \text{if } e < -d_e.
\end{cases} \]  

(14)

with \( d_e > 0 \) and \( 0 \leq \alpha_e \leq 1 \). Differing values of \( \alpha_e \) and \( d_e \) would reflect differing risk aversion on behalf of the economic agents.

Clearly the dynamical system (13) with (13b’) replacing (13b) still has the same local stability behaviour as previously. However the \( \alpha_e e \) component of \( \text{ref}(e) \) will hasten the return of trajectories back towards equilibrium in the situation of local instability as it reduces the effect of the unstable component of the eigenvalues. This analysis is confirmed in Figure 2.
which shows how the amplitude of the limit cycle decreases for various values of $\alpha_e$ and $d_e$. Indeed for extreme values of $\alpha_e$, $d_e$ (e.g. $\alpha_e = 0.25$, $d_e = 0.01$ in Figure 2b) the amplitude can be made very small and the dynamics appear to be stable.

4. **Maximum Likelihood Learning by Sophisticated Agents**

In this section it is more convenient to adapt a discrete time version of the model. Thus if $h$ denotes the time step we write the model as

$$e_{t+h} = e_t + h\gamma_e \left[ \phi(x_t + \frac{1}{\alpha_2} p_t) - e_t \right],$$  \hspace{1cm} (15a)

$$x_{t+h} = x_t + h\gamma_x \left[ \frac{(e_{t-h} - e_t) - x_t}{h} - x_t \right],$$  \hspace{1cm} (15b)

$$p_{t+h} = p_t + h\rho \left[ \frac{(\beta_2 + \alpha_2 \beta_3)}{\alpha_2} p_t + \beta_3 e_t \right].$$  \hspace{1cm} (15c)

This modelling of the foreign exchange market assumes that the sophisticated economic agents have complete knowledge of the foreign exchange market in that they know the speed of adjustment $\gamma_e$ and all the parameters of the foreign asset demand function $\phi$. They need this knowledge in order to be able to form the right-hand side of equation (15b).

In this section we assume that the sophisticated economic agents do not have full knowledge of these parameters and seek to learn them from observations of the evolution of the system. Furthermore these observations (i.e. of $e_t$, $p_t$) are noisy due to the incessant arrival of news in the markets for $e$ and $p$. Thus observed values for $e_t$ and $p_t$ are written

$$\tilde{e}_t = e_t + \sigma_e \sqrt{h} \xi_t^{(e)}$$  \hspace{1cm} (16a)

$$\tilde{p}_t = p_t + \sigma_p \sqrt{h} \xi_t^{(p)}$$  \hspace{1cm} (16b)

where $\xi_t^{(i)} \sim N(0, 1)$ ($i = e, p$) are independent standard normal variables.
We shall assume in turn that the economic agents seek to learn \( n \) (which amounts to learning the equilibrium of the system) and \( \gamma_e \) (which amounts to learning the market’s speed of adjustment to equilibrium).

Consider the case when the economic agents seek to learn the parameter \( n \). Let \( n^* \) be the "true" value of \( n \) (i.e. this is the value used by the foreign exchange market in equation (15a)), and \( n_t \) the forecast of \( n \) used by agents in period \( t \) in forming their expectations in equation (15b).

Thus the evolution of market prices are given by

\[
\ddot{e}_{t+h} = \ddot{e}_t + h \gamma_e [\Phi(x_t + \frac{1}{\alpha_2} \tilde{p}_t; n^*) - \ddot{e}_t] + \sigma_e \sqrt{h} \xi_t^{(e)},
\]

\[ (17a) \]

\[
x_{t+h} = x_t + h \gamma_x [\gamma_e \{\Phi(x_t + \frac{1}{\alpha_2} \tilde{p}_t; n_t) - \ddot{e}_t\} - x_t],
\]

\[ (17b) \]

\[
\ddot{p}_{t+h} = \ddot{p}_t + h \rho \left[ -\frac{(\beta_2 + \alpha_2 \beta_3)}{\alpha_2} \ddot{p}_t + \beta_3 \ddot{e}_t \right] + \sigma_p \sqrt{h} \xi_t^{(p)}.
\]

\[ (17c) \]

We now need to indicate the dependence of the function \( \Phi \) on \( n \), so it is now written \( \Phi(z; n) \). We place \( \sim \) above \( e \) and \( p \) to indicate that these have now become stochastic variables. In order to update their estimate of \( n_t \), the economic agents must estimate the evolution of \( e_t \), \( p_t \). This they do according to

\[
\dot{e}_{t+h}^{(est)} = \dot{e}_t + h \gamma_e [\Phi(x_t + \frac{1}{\alpha_2} \tilde{p}_t; n_t) - \ddot{e}_t],
\]

\[ (18a) \]

\[
x_{t+h} = x_t + h \gamma_x [\gamma_e \{\Phi(x_t + \frac{1}{\alpha_2} \tilde{p}_t; n_t) - \ddot{e}_t\} - x_t],
\]

\[ (18b) \]

\[
\dot{p}_{t+h}^{(est)} = \dot{p}_t + h \rho \left[ -\frac{(\beta_2 + \alpha_2 \beta_3)}{\alpha_2} \dot{p}_t + \beta_3 \ddot{e}_t \right].
\]

\[ (18c) \]

The economic agents are then able to form their estimation error vector
\[
\eta_t = \begin{bmatrix} \hat{e}_t \\ \hat{\rho}_t \end{bmatrix} - \begin{bmatrix} e_t^{(est)} \\ \rho_t^{(est)} \end{bmatrix}
\]

which they then use to update their estimate of \( n_t \). In calculating these updates we have used the algorithm proposed by Sidar (1976) which is outlined in Appendix 1.

This algorithm employs a maximum likelihood approach and updates \( n_t \) according to

\[
n_t = n_{t-h} + A^{-1} B \eta_t.
\]

where the matrices \( A \) and \( B \) are generated recursively in the way outlined in Appendix 1.

In the simulations reported below we have used the basic parameter set

\[
\begin{align*}
\alpha_2 &= -3, \quad \beta_2 = -1, \quad \beta_3 = 0.35, \quad n* = 0.05 \\
\gamma_e &= 3.33, \quad \gamma_x = 5, \quad \rho = 2.5 \\
h &= 0.001 \\
\sigma_e &= 0.1, \quad \sigma_p = 0.1.
\end{align*}
\]
For this set of parameters the model with no noise and no learning exhibits limit cycle behaviour. Figure 3 shows the evolution of \( n \), from an initial value within 10% of the true value, over a time interval which is 10 years on the time scale of the model. Whilst the agents' estimate of \( n \) spends long periods in the vicinity of the true value it also undergoes significant deviations from time to time. This behaviour may be due to the underlying limit cycle and the fact that the changing values of \( n \) in the difference equation for \( x \), tend to make the system's equilibrium point shift around so that the economic agents end up chasing a moving target. Figure 4 shows the evolution of exchange rate changes \((e_{t+h} - e_{t})\) over the 10 year period both with learning and without learning, using the same set of random noise terms. The exchange rate changes generated by the system with learning are clearly more volatile. We also see from Figure 4 that the underlying limit cycle is barely affected by the learning process. Figure 5 displays the histograms of exchange rate changes over the 10 year period with and without learning. Figure 6 shows the exchange rate changes over the first six months and it is interesting to observe that the simulated time series with learning exhibits some of the characteristics of observed time series, such as the clustering of large and small changes and a high frequency of outliers relative to the normal distribution.

Now suppose that the economic agents do not know \( \gamma_e \) and seek to learn it by the adaptive learning algorithm described in Appendix 2. We now use the same basic parameter set as in the simulations for the learning of \( n \). Figure 7 shows the evolution of \( \gamma_e^t \) over 10 years, the economic agents' estimate of \( \gamma_e \). The learning algorithm now converges to small fluctuations around the true value of \( \gamma_e \). This behaviour contrasts with the previous poorer convergence of learning of \( n \). The difference may be explained by the fact that economic agents' differing estimates of \( \gamma_e \) affect the speed of adjustment to the underlying attractor but not the "shape" of the attractor which is more determined by the asset demand function \( \phi \). Figure 8 displays exchange rate changes over the 10 years of the learning period and Figure 9 the corresponding histogram. Again we observe that the feedback effects of learning increase volatility.

5. **The Interaction of Sophisticated Agents and Naive Speculators**

In this system we shall assume that the foreign exchange market is populated by two distinct groups of economic agents. One group is the group of sophisticated agents who have incurred the set-up costs to engage in maximum likelihood learning. This group may be thought of as large institutions which specialise in foreign exchange transactions. The second group is assumed to consist of naive (or at least less sophisticated) traders who react primarily to recent changes in the exchange rate. This group would normally constitute a small part of the market, but could become a larger part from time to time.
We let $x_t^1$ and $x_t^2$ denote, respectively, the expected exchange rate change of the sophisticated and naive traders. We denote by $\theta$ the proportion of the market formed by the sophisticated traders, so that the market's expectation of exchange rate changes, $x_t$, is formed according to

$$x_t = \theta x_t^1 + (1 - \theta) x_t^2.$$  \hfill (21)

In order to focus the analysis on the learning by the sophisticated traders of just one unknown quantity we assume that at time $t$ these agents know or are able to observe $\theta$ and $x_t^2$. We focus on the case when the sophisticated traders are trying to learn the speed of market adjustment $\gamma_e$. Thus with $x_t$ given by equation (21) the system dynamics now evolve according to

$$\bar{e}_{t+h} = \bar{e}_t + h \gamma^e \left[ \phi(x_t + \frac{1}{\alpha_2} \bar{\rho}_t) - \bar{e}_t \right] + \sigma_e \sqrt{n(t)}, \tag{22a}$$

$$x_{t+h}^1 = x_t^1 + h \gamma_{x^1} \left[ \phi(x_t + \frac{1}{\alpha_2} \bar{\rho}_t) - \bar{e}_t \right] - x_t^1, \tag{22b}$$

$$x_{t+h}^2 = x_t^2 + h \gamma_{x^2} \left[ \frac{(\bar{e}_t - \bar{e}_{t-h})}{h} - x_t^2 \right], \tag{22c}$$

$$\bar{\rho}_{t+h} = \bar{\rho}_t + h \rho \left[ - (\beta_2 + \frac{\alpha_2 \beta_3}{\alpha_2}) \bar{\rho}_t + \beta_3 \bar{e}_t \right] + \sigma_\rho \sqrt{n(t)} \left( 1 \right). \tag{22d}$$

We allow the sophisticated and naive traders to have different speeds of expectations adjustment, $\gamma_{x^1}$ and $\gamma_{x^2}$ respectively. In equation (22c) the naive traders are adjusting expectations to the most recently observed rate of exchange rate change. The necessary slight adjustments to the sophisticated agents' learning algorithm are described in Appendix 3. Note also that we allow the two groups to have differing speeds of expectations adjustment $\gamma_{x^1}$ and $\gamma_{x^2}$.

Figure 10 shows the evolution of $\gamma_e$ for $\theta = 0.8$ and differing values of $\gamma_{x^2}$. The more rapid speed of expectations adjustment on the part of naive traders brings about a slower convergence of learning. Figure 11 shows the effect of a marked increase in the proportion of the naive traders, which we see tends to destabilise the learning procedure.
6. **Conclusion**

Conclusions at this stage are tentative as more work needs to be done on the econometric analysis of simulated time series. However the attempt by sophisticated traders to learn in a stochastic nonlinear environment seems to markedly increase volatility of exchange rate changes. This effect on the volatility is more pronounced than that of the interaction of the nonlinear and stochastic elements in a no-learning environment. The introduction of naive traders makes the convergence of the learning by sophisticated traders more difficult however the effects on the volatility of exchange rate changes is not so clear cut and requires further econometric analysis.
Appendix 1 The Recursive Identification Algorithm

Introduce the following vector notation

\[
X_t = \begin{pmatrix} e_t \\ x_t \\ \rho_t \end{pmatrix}, \quad F(X; n*, n) = \begin{pmatrix} \gamma_e \{ \phi(x + \frac{1}{\alpha_2} p; n*) - e \} \\ \gamma_x \{ \phi(x + \frac{1}{\alpha_2} p; n) - e \} - x \\ \rho - \frac{\beta_2 + \alpha_2 \beta_3}{\alpha_2} p + \beta_3 e \end{pmatrix}
\]

\[
V_t = \begin{pmatrix} \xi_t^{(e)} \\ \xi_t^{(p)} \\ \bar{e}_t \end{pmatrix}, \quad E[V_t] = 0, \quad E[V_t V_t^\top] = R = \begin{pmatrix} \sigma_e^2 & \sigma_{ep} \\ \sigma_{ep} & \sigma_p^2 \end{pmatrix}
\]

\[
Y_t = \begin{pmatrix} e_t \\ \rho_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} X_t = C X_t
\]

The market prices are formed by the system

\[
X_{t+h} = X_t + h F(X_t; n*, n_t) + V_t
\]

and the economic agents' estimation of the evolution of the prices over the next time period is

\[
X_{t+h}^{est} = X_t + h F(X_t; n_t, n_t)
\]

At each \( t \) the economic agents can form the estimation error

\[
\eta_t = X_t - X_t^{est}
\]

To conform to Sidar's notation let

\[
P_t = \eta_t.
\]

Then the algorithm for updating \( P_t \) is
\[ P_t = P_{t-1} + A^{-1}B \eta_t \]

where
\[ B = S_t^T C^T \hat{R}_{t-1}^{-1} \]
\[ A = S_t^T C^T \hat{R}_{t-1}^{-1} C S_t \]
\[ \hat{R}_{t-1} = \frac{t}{t-h} \hat{R}_{t-h} - \frac{th}{(t-h)^2} \hat{R}_{t-h} \eta_t \eta_t^T \hat{R}_{t-h} \]
\[ S_t = S_{t-h} + h(F_{X}^{t-h}S_{t-h} + F_{P}^{t-h}) \]

where
\[ F_X^t = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \frac{\partial F_1}{\partial X_3} \\ \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \frac{\partial F_2}{\partial X_3} \\ \frac{\partial F_3}{\partial X_1} & \frac{\partial F_3}{\partial X_2} & \frac{\partial F_3}{\partial X_3} \end{bmatrix} \]

\[ = \begin{bmatrix} -\gamma_e & \gamma_e \phi'(x_t + \frac{1}{\alpha_2} p_t; \eta_t) & \frac{\gamma_e}{\alpha_2} \phi'(x_t + \frac{1}{\alpha_2} p_t; \eta_t) \\ -\gamma_x \gamma_e & \gamma_x \gamma_e \phi'(x_t + \frac{1}{\alpha_2} p_t; \eta_t) & \gamma_x \frac{\gamma_e}{\alpha_2} \phi'(x_t + \frac{1}{\alpha_2} p_t; \eta_t) \\ \rho \beta_3 & 0 & -\rho (\beta_2 + \alpha_2 \beta_3) \end{bmatrix} \]

and
\[ F_p^f = \begin{bmatrix} \frac{\partial F_1}{\partial \mu} \\ \frac{\partial F_2}{\partial \mu} \\ \frac{\partial F_3}{\partial \mu} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \mu} \left( x_t + \frac{1}{\alpha_2} \mu; n_t \right) \\ \frac{\partial}{\partial n} \left( x_t + \frac{1}{\alpha_2} \mu; n_t \right) \\ \frac{\partial}{\partial \mu} \left( x_t + \frac{1}{\alpha_2} \mu; n_t \right) \\ \frac{\partial}{\partial n} \left( x_t + \frac{1}{\alpha_2} \mu; n_t \right) \end{bmatrix} \]

The matrix \( S \) is the sensitivity matrix i.e.

\[ S = \begin{bmatrix} \frac{\partial X_1}{\partial \mu} \\ \frac{\partial X_2}{\partial \mu} \\ \frac{\partial X_3}{\partial \mu} \end{bmatrix} \]
Appendix 2 Learning $\gamma_e$

We employ the same notation as in Appendix 1 except that we now emphasise the dependence of the right-hand side of the dynamical system upon $\gamma_e$ by writing

$$F (X; \gamma^*, \gamma) = \begin{bmatrix} 
\gamma^* \left[ \phi \left( x + \frac{1}{\alpha_2} p \right) - e \right] \\
\gamma_x \left[ \gamma \left\{ \phi \left( x + \frac{1}{\alpha_2} p \right) - e \right\} - x \right] \\
\rho \left[ - \frac{(\beta_2 + \alpha_2 \beta_3)}{\alpha_2} p + \beta_3 e \right] 
\end{bmatrix}$$

We use $\gamma^*$ to denote the true value of $\gamma_e$ and $\gamma_t$ the economic agents' estimate of $\gamma_e$ in time period $t$. The market prices are now formed by the system

$$X_{t+h} = X_t + h F (X_t; \gamma^*, \gamma_t) + V_t,$$

and the economic agents' estimation of the evolution of prices over the next time period is

$$X_{t+h}^{est} = X_t + h F (X_t; \gamma_t, \gamma_t).$$

with $X_t$ and $X_t^{est}$ so defined and setting

$$p_t = \gamma_t,$$

the notation of Appendix 1 now describes the estimation procedure for $\gamma_t$. There are two further changes to be made. Firstly, in the calculation of $F_x^t$, $n_t$ is replaced by $n^*$ and $\gamma_e$ by $\gamma_t$. Secondly the vector $F_p^t$ is given by

$$F_p^t = \begin{bmatrix} 
\phi \left( x_t + \frac{1}{\alpha_2} p_t \right) - e_t \\
\gamma_x \phi \left( x_t + \frac{1}{\alpha_2} p_t \right) - e_t \\
0 
\end{bmatrix}.$$
Appendix 3 Learning $\gamma_e$ in the Presence of Naive Traders

In terms of the notation of Appendix 2 we define

$$X_t = \begin{bmatrix} e_t \\ x^1_t \\ p_t \\ x^2_t \end{bmatrix}, \quad F(X; \gamma^*, \gamma) = \begin{bmatrix} \gamma^* [\phi(x + \frac{1}{\alpha_2} p) - e] \\ \gamma_x [\phi(x + \frac{1}{\alpha_2} p) - e] - x] \\ \rho \left[ - \frac{\beta_2 + \alpha_2 \beta_3}{\alpha_2} p + \beta_3 e \right] \\ \gamma_{x_2} \left[ \frac{(e_t - e_t - h)}{h} - x_2 \right] \end{bmatrix}$$

The matrix $F_{X^t}$ will have the form

$$F_{X^t} = \begin{bmatrix} \gamma^*(1 - \theta) \phi^1(x + \frac{1}{\alpha_2} p) \\ F_{X^t}^{(old)} \\ \vdots \\ \gamma_{x_2} \gamma_x (1 - \theta) \phi^1(x + \frac{1}{\alpha_2} p) \\ 0 \\ \vdots \\ \vdots \\ \gamma_{x_2} \\ h \end{bmatrix}$$

where $F_{X^t}^{(old)}$ is the $F_{X^t}$ used in Appendix 2, but with $\gamma_{x_1}$ replacing $\gamma_x$.

Finally

$$F_{p^t} = \begin{bmatrix} F_{p^t}^{(old)} \\ 0 \end{bmatrix}$$
where $F_p^t(\text{old})$ is the $F_p^t$ used in Appendix 2 with $\gamma_{x_{1}}$ replacing $\gamma_{x}$. 
References


Figure 2 - Naive Learning $d_\pi = 0.1$

$\alpha_e = 1$ blue, $\alpha_e = 0.75$ green, $\alpha_e = 0.50$ red, $\alpha_e = 0.25$ magenta

Figure 3
Figure 4 - Change of Exchange rate

Figure 5a - $n$, Learning
Histogram of Change of Exchange Rate With Noise
Figure 6b - No learning with Noise Model
Histogram of Change of Exchange Rate Model

Figure 5a - Change of Exchange Rate: Black - Learning n, (L) + Noise(N), Magenta - No Learning(NL) + N, Blue - L + No Noise(NN), Yellow - NL + NN
Figure 6b - Change of Exchange Rate: Black - No Learning (NL) + Noise (N), Magenta - Stable Linear System + N, Yellow - NL + No Noise

Figure 7 - Learning $\tau_1$, Plot of $\gamma_c$
Figure 8 - Learning $\gamma_c$, Black - Interaction(I) + Noise(N), Magenta - No Interaction(NI) + N, Blue - I + No Noise(NN), Yellow - NI + NN

Figure 9 - Learning Parameter $\tau$
Change of Exchange Rate with Noise
Fig 10 - Learning $\gamma_e$ in Presence of Naive Traders, Black - $\theta = 1$, Blue - $\theta = 0.8$, $\gamma_{s,2} = 0.2$, Red - $\theta = 0.8$ $\gamma_{s,2} = 10$

Black - Learning $n_t$ + Noise, Magenta - No Learning + Noise,
Blue - Learning $n_t$ + No Noise, Yellow - No Learning + No Noise

Figure 1 - Learning in the Presence of Naive Traders
Black - $\theta = 1$, Magenta - $\theta = 0.5$, $\gamma_{s,2} = 5$