Approximating Heath-Jarrow-Morton Non-Markovian Term Structure of Interest Rate Models with Markovian Systems

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Abstract:

We consider a Heath-Jarrow-Morton models for the term structure of interest rates in which the forward rate volatility is a function of the instantaneous spot rate of interest, a set of discrete forward rates and time to maturity of the bond. We show how the stochastic dynamics may be expressed as a system of Markovian stochastic differential equations. We obtain the partial differential equation which allows the pricing of contingent claims in this framework.

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Acknowledgment: The authors are indebted to Peter Ritchken for suggesting to them that the type of volatility function specified in this paper should lead to a Markovian representation. The usual caveat applies.
1. Introduction

Approaches to modelling the term structure of interest rates in continuous time may be broadly described in terms of either the equilibrium approach or the no-arbitrage approach even though some early models include concepts from both approaches.

The equilibrium approach is now based primarily on the work of Cox, Ingersoll and Ross (1985). This approach begins with a description of the underlying economy. First, assumptions are made about the stochastic evolution of exogenous factors or state variables in the economy and about the preferences of individual investors. Then a stochastic intertemporal optimising framework is employed to endogenously derive the interest rate and the price of contingent claims.

The no-arbitrage approach starts with assumptions about the stochastic evolution of one or more underlying factors, usually interest rates. Bond prices are assumed to be functions of these driving stochastic processes. The prices of contingent claims are then derived by imposing the condition that there are no arbitrage opportunities in the economy between bonds of different maturity. Vasicek's (1977) model is an example of a single-factor no-arbitrage model in which the whole term structure depends on a single stochastic variable or factor, in this case the short interest rate. It employs an arbitrage argument in relation to the expected return on bonds of differing maturity. The Brennan and Schwartz (1979) approach is based on two factors, the shortest and the longest maturity yields and also uses a form of arbitrage argument in its approach.

The early no-arbitrage models contained terms related to the market price of interest rate risk, which is dependent upon unobservable investor preferences. These models also had the shortcoming of not being consistent with the currently observed yields. A major advance in the modelling of the term structure of interest rates was made by Ho and Lee (1986) who, in a binomial framework using no-arbitrage arguments, derived a model which does not depend upon the market price of interest rate risk. This feature was achieved by imposing the condition that the model is compatible with the currently observed yield curve, which has impounded in it investor preferences and the market price of interest rate risk. This approach was generalised to the continuous time framework by Heath, Jarrow and Morton (HJM, 1992) which now provides the most general methodology for term structure of interest rate models. The HJM approach uses
as the driving stochastic dynamic variable forward rates whose evolution is dependent on a specified volatility function.

The crucial input to the HJM model is the volatility function of the forward interest rate and the most important modelling decision is the form of this volatility function. Rebonato (1998, page 316) makes the point that there is not “the HJM” model but a class of HJM models each characterised by a specification of this volatility function. Ritchken and Sankarasubramanian (1995a) also investigate the possible effect of this specification on estimated option prices. Hull and White (1990) develop a methodology to eliminate the market price of risk by imposing consistency with the currently observed yield curve. Pelsser (1996) indicates a more systematic procedure which works well for a certain class of models. It is now understood that these approaches can be seen to be equivalent to assuming certain forms for the forward rate volatility in the HJM framework. See Bhar and Chiarella (1997a), Chiarella and El-Hassan (1996) and Chiarella and Kwon (1999a).

In the arbitrage-free approach of HJM the dependence of the drift on the volatility function is explicitly recognised. A consequence of this is that the spot rate process is history dependent and results, under the most general specification, in a non-Markovian structure. Several authors have investigated the conditions under which the spot rate process may be made Markovian. Such studies include, for example, Cheyette (1992), Carverhill (1994), Ritchken and Sankarasubramanian (1995b), Jeffrey (1995) and Bhar and Chiarella (1997a). Although these studies differ in details of their implementation, the general nature of the volatility specification is given by,

\[ \sigma(t,T,r(t)) = \left[ a_0 + a_1(T-t) + \ldots + a_n(T-t)^n \right] h(r(t)) e^{-\lambda(T-t)}, \quad 0 \leq t \leq T \quad (1) \]

where \( h(r(t)) \) is a function of the spot interest rate, and the \( a_i \) and \( \lambda \) are constant coefficients. The effect of such a specification is that the system becomes Markovian but with respect to an augmented set of state variables. Bhar and Chiarella (1997a) demonstrate, using the general specification as in equation (1), the number of additional state variables needed and the precise nature of these subsidiary variables. They also express such a higher dimensional system in state space form, which can then be estimated using the extended Kalman filter algorithm. Inui and
Kijima (1998) consider multiple noise terms with each volatility function being of the form (1), with \( a_1 = \ldots = a_n = 0 \) and \( \lambda \) possibly being time dependent in a particular way.

In this paper we extend Bhar and Chiarella (1997a) to further generalise the form of the volatility function to include forward interest rates. In particular we work with the specification,

\[
\sigma(t, T, r(t), f(t, \tau)) = g[r(t), f(t, \tau)] e^{-\lambda(T-t)}, \quad 0 \leq t \leq \tau < T
\]

and determine the additional state variables necessary to make the system Markovian although with a higher dimension. Here \( g(r(t), f(t, \tau)) \) is a function of the spot interest rate, \( r(t) \), and of the forward interest rate, \( f(t, \tau) \) of a fixed maturity \( \tau \). For example, \( f(t, \tau) \) could be some long forward rate. The intuition behind such a specification is that not only the spot interest rate but also a fixed maturity forward interest rate influence the evolution of the term structure. The particular forward rate to be used may depend on the application under consideration. This approach may be considered to be equivalent in some sense, within the HJM framework, to the Brennan and Schwartz (1979) model where a short rate and a long rate are used to explain the evolution of the term structure.

The cited earlier works obtain Markovian representations by an expansion of the state space as we have already mentioned. However the additional state variables have never been given any economic interpretation. A distinguishing feature of our analysis is that we demonstrate how to express the additional state variable in terms of \( r(t) \) and \( f(t, \tau) \). Thus the Markovian representation which initially appears as a three dimensional stochastic dynamic system can be reduced to a two dimensional one. Furthermore, it turns out that the evolution of the forward rate curve is determined entirely by the evolution of \( r(t) \) and \( f(t, \tau) \).

In the next section we describe the basic HJM framework and show how under the volatility specification of equation (2) the driving non-Markovian stochastic dynamics may be reduced to Markovian form by the introduction of an additional state variable. The stochastic dynamics are then being driven by a three-dimensional Markovian system. In section 3, we show the additional state variable may be expressed in terms of the rates \( r(t) \) and \( f(t, \tau) \), thus reducing the driving stochastic dynamics to a two-dimensional Markovian system. In section 4 we show
how an explicit expression for bond prices may be obtained. We also derive the underlying pricing partial differential operator which turns out to involve two spatial variables. In section 5 we discuss the pricing of bond options within the framework we have adopted. Section 6 concludes. Many of the technical derivations are gathered in a set of appendices.

2. The Forward Rate Process

2.1 The HJM Framework

We recall that the starting point of the one factor HJM (1992) model of the term structure of interest rates is the stochastic integral equation for the forward rate

\[
f(t, T) = f(0, T) + \int_0^t \alpha(u, T, \cdot) du + \int_0^t \sigma(u, T, \cdot) dW(u), \quad (0 \leq t \leq T).
\]

Here \(f(t, T)\) is the forward rate at time \(t\) applicable to time \(T(>t)\). The noise terms \(dW(u)\) are the increments of a standard Wiener process generated by a probability measure \(Q\). These are the shock terms arising from the underlying factor. The functions \(\alpha(u, T, \cdot)\) and \(\sigma(u, T, \cdot)\) are respectively the instantaneous drift and volatility functions associated with the noise term. The third argument (\(\cdot\)) indicates possible dependence on other state variables, typically such dependence could be on \(f(t, T)\) itself or on \(r(t)\), the instantaneous spot rate of interest both at time \(t\).

HJM show that the absence of riskless arbitrage opportunities implies that the drift term cannot be chosen arbitrarily but rather will be a function of the volatility function, \(\sigma(\cdot)\), and the market price of interest rate risk, \(\phi(t)\). This relationship is given by,

\[
\alpha(t, T) = -\sigma(t, T, \cdot) \left[ \phi(t) - \int_t^T \sigma(s, t, \cdot) ds \right].
\]

Furthermore, by an application of Girsanov’s theorem the dependence on the market price of interest rate risk can be absorbed into an equivalent probability measure. The stochastic integral equation for the forward rate then becomes,

\[
f(t, T) = f(0, T) + \int_0^t \sigma(u, T, \cdot) \int_u^T \sigma(u, s, \cdot) du ds + \int_0^t \sigma(u, T, \cdot) d\tilde{W}(u).
\]
Here, $\tilde{W}(u)$ is the Wiener process generated by an equivalent martingale probability measure $\tilde{Q}$. The probability measures $Q$ and $\tilde{Q}$ are related, via the Radon-Nikodym derivative as explained in HJM.

From equation (5) we obtain the stochastic integral equation for the spot interest rate as $r(t) = f(t,t)$ which under $\tilde{Q}$ is given by,

$$r(t) = f(0,t) + \int_0^t \sigma(u,t_s) \int_0^t \sigma(u,s_s) dsdu + \int_0^t \sigma(u,t_s) d\tilde{W}(u). \quad (6)$$

As a stochastic differential equation (6) becomes,

$$dr(t) = \left[ f_2(0,t) + \frac{\partial}{\partial t} \int_0^t \sigma(u,t_s) \int_0^t \sigma(u,s_s) dsdu + \int_0^t \sigma_2(u,t_s) d\tilde{W}(u) \right] dt + \sigma(t,t_s) d\tilde{W}(t), \quad (7)$$

where $f_2(0,t)$ denotes the partial derivative of $f(0,t)$ with respect to the second argument. The non-Markovian nature of the stochastic dynamics of the system that we are considering stems from the integral terms of the drift coefficient. These integral terms depend on the entire history of the process up to time $t$.

Our aim now is to express the equations (5) and (6) under the volatility specification of equation (2) as a Markovian system of stochastic differential equations (SDE). With such a volatility specification we can rewrite the stochastic integral equations (5) and (6) as

$$f(t,T) = f(0,T) + \int_0^t \sigma(u,T,r(u),f(u,\tau)) \int_0^T \sigma(u,s,r(u),f(u,\tau)) dsdu$$

$$+ \int_0^t \sigma(u,T,r(u),f(u,\tau)) d\tilde{W}(u), \quad (8)$$

and,

$$r(t) = f(0,t) + \int_0^t \sigma(u,t,r(u),f(u,\tau)) \int_0^t \sigma(u,s,r(u),f(u,\tau)) dsdu$$

$$+ \int_0^t \sigma(u,t,r(u),f(u,\tau)) d\tilde{W}(u). \quad (9)$$
2.2 \textbf{Reduction to Markovian Form}

In order to express the stochastic integral equations (8) and (9) as stochastic differential equations, we need the stochastic differentials of the second and the third terms of these equations. These are evaluated in Appendix A, and for (8) the resulting SDE is,

\[
df(t, T) = \left[ \sigma(t, T, r(t), f(t, \tau))^2 \left( \frac{e^{\lambda(T-t)} - 1}{\lambda} \right) \right] dt + \sigma(t, T, r(t), f(t, \tau))d\tilde{W}(t). \tag{10}
\]

Similarly, the SDE expression for the stochastic integral equation (9) is given by

\[
dr(t) = \left[ f_2(0, t) + \int_{0}^{t} \left( -\lambda \sigma(u, t, r(u), f(u, \tau)) \int_{u}^{t} \sigma(u, s, r(u), f(u, \tau))ds + \right. \right.

\left. \left. \sigma(u, t, r(u), f(u, \tau))^2 du - \lambda \int_{0}^{t} \sigma(u, t, r(u), f(u, \tau))d\tilde{W}(u) \right) \right] dt + \sigma(t, t, r(t), f(t, \tau))d\tilde{W}(t). \tag{11}
\]

The calculations of the differentials required to obtain the second and the third terms on the right hand side of equation (11) are given in Appendix B. We note from equation (9) that

\[
\int_{0}^{t} \sigma(u, t, r(u), f(u, \tau))\int_{u}^{t} \sigma(u, s, r(u), f(u, \tau))dsdu

+ \int_{0}^{t} \sigma(u, t, r(u), f(u, \tau))d\tilde{W}(u) = r(t) - f(0, t). \tag{12}
\]

Then the stochastic differential equation (11) may be simplified to

\[
dr(t) = [f_2(0, t) + \lambda f(0, t) + \psi(t) - \lambda r(t)]dt + \sigma(t, t, r(t), f(t, \tau))d\tilde{W}(t), \tag{13}
\]

where we define the subsidiary variable
\[ \psi(t) = \int_0^t \sigma(u, r(u), f(u, \tau))^2 du. \] (14)

It is a straightforward matter to show (see appendix C) that \( \psi(t) \) satisfies the stochastic differential equation

\[ d\psi(t) = [\sigma(t, t, r(t), f(t, \tau))^2 - 2\lambda\psi(t)] dt. \] (15)

We have now reduced the non-Markovian stochastic dynamics to a three dimensional Markovian stochastic dynamical system consisting of the stochastic differential equation for the discrete forward rate, namely,

\[ df(t, \tau) = \sigma(t, \tau, r(t), f(t, \tau))^2 \frac{[e^{\lambda(t-\tau)} - 1]}{\lambda} dt + \sigma(t, \tau, r(t), f(t, \tau)) d\tilde{W}(t), \] (16)

and the two stochastic differential equations (13) and (15) for \( r(t) \) and \( \psi(t) \) respectively.

Finally we recall that the dynamics of the forward rate to any maturity \( T \), \( f(t, T) \) is given by equation (8) and so is determined once \( r(t) \) and \( f(t, \tau) \) are determined. These latter quantities are driven by the three SDEs (13), (15), (16) which together form the Markovian representation.

The price of any derivative instrument would then have to depend on \( r(t) \) and \( f(t, \tau) \). Thus a bond of maturity \( T \) would have a price at time \( t \) denoted by, \( P(t, T, r(t), f(t, \tau)) \), and this price is also driven by the three-dimensional Markovian SDE system referred to above.

3. **Interpreting the Subsidiary Variable \( \psi(t) \)**

The subsidiary variable \( \psi(t) \) defined in equation (14) plays a central role in allowing us to transform the original non-Markovian dynamics to Markovian form. Similar subsidiary variables appear in the reduction to Markovian forms of Cheyette (1992), Ritchken and Sankarasubramanian (1995b), Bhar and Chiarella (1997a), Inui and Kijima (1998) and Chiarella and Kwon (1999b).
It is clear from equation (14) that \( \psi(t) \) may be interpreted as a variable summarising the path history of the forward rate volatility. Similar interpretations have been given by the other cited authors. However, it would perhaps be more satisfying to relate \( \psi(t) \) to the market rates \( r(t) \) and \( f(t, \tau) \). Indeed it turns out that such a relationship does exist for the forward rate volatility function assumed in equation (2).

**Proposition 1**

The subsidiary integrated square variance quantity \( \psi(t) \) defined in equation (14) is related to the rates \( r(t) \) and \( f(t, \tau) \) via

\[
\psi(t) = \lambda \alpha(t, \tau) [r(t) - f(0, t)] - \lambda e^{-\lambda(t-\tau)} \alpha(t, \tau) [f(t, \tau) - f(0, \tau)]
\]

where \( \alpha(t, \tau) = e^{-\lambda t} / (e^{-\lambda \tau} - e^{-\lambda t}) \) (17)

**Proof**

See appendix D.

An important consequence of Proposition 1 is that it allows us to reduce by one the dimension of the stochastic dynamic system (13),(15),(16) to the two-dimensional one consisting of the stochastic differential equations (13) and (16) with \( \psi(t) \) begin defined by equation (17).

This reduction in dimension is quite significant if we seek to solve for derivative prices in this framework by use of partial differential equations or lattice based methods as in section 6, since then we need only deal with two rather than three spatial variables in the partial differential operator. The reduction is less significant, though still useful, when using Monte Carlo simulation. This is so since Monte Carlo simulation requires the simulation of the one Wiener increment, \( d\hat{W}(t) \). The generation of \( \psi(t) \) by equation (17) rather than discretising equation (15) should lead to some computational efficiency.

A consequence of proposition 1 is that we are able to express the forward rate to any maturity \( T \) in terms of the two rates \( r(t) \) and \( f(t, \tau) \).
Proposition 2

The forward rate $f(t,T)$ to any maturity $T$ is given by

$$f(t,T) - f(0,T) = \frac{\alpha(t,\tau)}{\alpha(T,\tau)} [f(t,\tau) - f(0,\tau)] + \left( 1 + \frac{\alpha(t,\tau)}{\alpha(T,\tau)} \right) [r(t) - f(0,t)].$$

Proof

See appendix E. ■

We note that in the special case when $\sigma$ does not depend explicitly on $f(t,\tau)$ we may still carry out the set of manipulations leading to the result (17). Now of course the choice of the particular discrete tenor forward rate is somewhat arbitrary, but once a choice is made, the representation (17) can be obtained. This special case corresponds to the class of volatility functions considered by Ritchken and Sankarasarubrananian (1995b) and so we see in this way how it is possible to give an economic interpretation to the subsidiary variable that arises in their analysis.

4. The Term Structure of Interest Rates

We recall the HJM approach of defining a money account,

$$A(t) = \exp \left( \int_0^t r(y)dy \right),$$

and showing that the relative bond price

$$Z(t,T) = \frac{P(t,T)}{A(t)},$$

is a martingale, so that the bond price can be written\(^1\)

$$P(t,T) = \tilde{E}_t \left( \frac{A(t)}{A(T)} \right) = \tilde{E}_t \left( \exp \left\{ - \int_t^T r(y)dy \right\} \right).$$

\(^1\) See HJM (1992) or, Bhar and Chiarella (1997a) for a simplified version.
Here, $\pi$, is the expectation taken with respect to the probability distribution generated by the stochastic differential system (13) and (16). We use
\[
\pi(r(t^*), f(t^*, \tau) | r(t), f(t, \tau)),
\]
(21)
to denote the transition probability density function between $t$ and $t^*$ ($t \leq t^*$). This quantity satisfies the Kolmogorov backward partial differential equation (see Oksendal (1992) for a discussion of the Kolmogorov backward equation for a multi-dimensional diffusion process), which for our case is given by,
\[
K\pi + \frac{\partial \pi}{\partial t} = 0,
\]
(22)
where the operator $K$ is the infinitesimal generator of the diffusion process for $f(t, \tau), r(t)$ driven by the stochastic differential equations (13) and (16). It turns out that $K$ is given by \(^2\) (see appendix F),
\[
K\pi \equiv \sigma_i^2 \left( \frac{e^{\lambda(t-t)}}{\lambda} \right) \frac{\partial \pi}{\partial f} + \left[ f_i(0, t) - \lambda f(0, t) + \psi - \lambda r \right] \frac{\partial \pi}{\partial r} + \\
\frac{1}{2} \sigma_i^2 \frac{\partial^2 \pi}{\partial f^2} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 \pi}{\partial r^2} + \sigma_i \sigma_r \frac{\partial^2 \pi}{\partial f \partial r}.
\]
(23)
By application of the Feynman-Kac formula to equation (24) we find that the bond price $P(t, T, r, f)$ satisfies the partial differential equation,
\[
KP - rP + \frac{\partial P}{\partial t} = 0,
\]
(24)
subject to the terminal condition
\[
P(T, T, r, f) = 1,
\]
(25)
and the boundary conditions
\[^2\] In order to alleviate the notation in describing the operator $K$ we set
\[
\sigma_i(t) \equiv \sigma(t, \tau, r(t), f(t, \tau)) \text{ and } \sigma_r(t) \equiv (t, t, r(t), f(t, \tau))
\]
\[ P(t,T,\infty, f) = 0, \quad (f \geq 0), \]
\[ P(t,T,r,\infty) = 0, \quad (r \geq 0). \]

The further boundary conditions \( P(t,T,0,f) \) and \( P(t,T,r,0) \) may be obtained by an extrapolation procedure to be discussed in section 6 below.

Note that in subsequent discussion we set
\[ D(t) = f_2(0,t) + \lambda f(0,t). \tag{26} \]

A consequence of proposition 2 is that it turns out to be possible to obtain an analytical expression for the bond price. In fact we may state the following proposition:-

**Proposition 3**

The price of bonds driven by the Markovian stochastic differential equation system (13) and (16) can be expressed as
\[ P(t,T,r,f) = \frac{P(0,T)}{P(0,t)} \exp \left\{ -\beta(t,T) \left[ r(t) - f(0,t) \right] - \frac{1}{2} \beta(t,T)^2 \psi(t) \right\}. \tag{27} \]

where
\[ \beta(t,T) = \frac{1}{\lambda} \left[ 1 - e^{-\lambda T} \right]. \]

and \( \psi(t) \) is defined in equation (17).

**Proof**

See appendix G. ■

The bond pricing equation (31) has precisely the same form\(^3\) as the one derived by Ritchken and Sankarasubramanian (1995a) who (in current notation) assumed a form for the volatility function in equation (2) which is independent of the forward rate \( f(t,\tau) \). In fact the results in propositions 2 and 3 can be considerably generalised. Chiarella and Kwon (1999b) have shown that (27) holds in precisely the same form even when the forward rate volatility depends on a set of

\(^3\) It is of course possible to express the exponent in equation (27) as a linear combination of \([r(t) - f(0,t)]\) and \([f(t,\tau) - f(0,\tau)]\). We give the form shown in the proposition in order to allow easier comparison with the work of Ritchken and Sankarasubramanian (1995a).
discrete forward rates \( f(t, \tau_1), f(t, \tau_2), \ldots, f(t, \tau_s) \) where \( t \leq \tau_1 < \tau_2 < \ldots < \tau_s \leq T \). Of course, under these different specifications the history variable \( \psi(t) \) will evolve differently but the functional relationship remains the same.

5. Pricing European Bond Options

Consider an option written on the bond of maturity \( T \). We suppose the option matures at time \( T_c < T \) and denote its price by \( C(t, T, r, f) \). This price satisfies the partial differential equation

\[
KC - rC + \frac{\partial C}{\partial t} = 0 \quad (0 \leq t \leq T_c).
\]  

(28)

If we are dealing with a European call option with strike price \( E \) then the terminal condition for equation (32) is

\[
C(T_c, T, r, f) = (P(T_c, T, r, f) - E)^+.
\]  

(29)

The boundary conditions at infinity are

\[
C(t, T, \infty, f) = 0, \quad f \geq 0,
\]

\[
C(t, T, r, \infty) = 0, \quad r \geq 0.
\]

We recall that the bond prices at option maturity for any given values of \( r(T_c), f(T_c, \tau) \) can be obtained directly from equation (27) without the need to solve the bond pricing partial differential equation (24).

In section 6 we discuss the solution of the partial differential equation (31) by means of the alternating directions implicit (ADI) method.

An alternative approach to pricing the European option is to use the result (also derived by HJM) that

\[
C(t, T, r, f) = \tilde{E}_t \left[ e^{-\int_0^T r(\tau) d\tau} (P(T_c, T, r(T_c), f(T_c, \tau)) - E)^+ \right].
\]  

(30)
The expectation in (30) could be approximated by simulating an appropriate number of times the stochastic differential equation system (13) and (16) from $t$ to $T_c$.

6. Conclusions

We have considered option pricing in the HJM framework with a forward rate volatility function depending upon time to maturity, the instantaneous spot rate of interest and a fixed forward rate. We have demonstrated how this specification allowed a two-dimensional Markovian representation of the stochastic dynamics. It thus becomes possible to price interest rate derivatives using partial differential equation techniques. Despite the fact that we are dealing with a partial differential equation with two-spatial dimensions the approach should be feasible as it is possible to obtain an explicit expression for the price of the underlying bond.

A number of directions for future research suggest themselves. First, it should be possible to further generalise the volatility function to allow for a number of finite-tenor forward rates. Second, it is also necessary to undertake some empirical studies to determine when the types of volatility functions discussed in this paper fit market conditions. Some initial steps in this direction have been taken by Bhar and Chiarella (1997b).
Appendix A (Derivation of equation 10)

To take the differential of the first integral term in equation (8), we need to evaluate,

\[
\frac{\partial}{\partial t} \int_0^t \sigma(u, T, r(u), f(u, \tau)) \int_u^T \sigma(u, s, r(u), f(u, \tau)) ds du
\]

\[
= \sigma(t, T, r(t), f(t, \tau)) \int_t^T \sigma(t, s, r(t), f(t, \tau)) ds
\]

\[
= \sigma(t, T, r(t), f(t, \tau)) g(r(t), f(t, \tau)) \int_t^T e^{-\lambda(t-s)} ds
\]

\[
= \sigma(t, T, r(t), f(t, \tau))^2 \left[ \frac{e^{\lambda(t-t)} - 1}{\lambda} \right].
\]

(A1)

The differential of the stochastic integral term is simply

\[
d \int_0^t \sigma(u, T, r(u), f(u, \tau)) d\tilde{W}(u) = \sigma(t, T, r(t), f(t, \tau)) d\tilde{W}(t).
\]

(A2)

With these two results we readily obtain the stochastic differential equation (10).

Appendix B (Derivation of equation 11)

Consider the time derivative of the first integral term of equation (9)

\[
\frac{\partial}{\partial t} \int_0^t \sigma(u, t, r(u), f(u, \tau)) \int_u^T \sigma(u, s, r(u), f(u, \tau)) ds du
\]

\[
= \int_0^t \left\{ \sigma_z(u, t, r(u), f(u, \tau)) \int_u^T \sigma(u, s, r(u), f(u, \tau)) ds + \sigma(u, t, r(u), f(u, \tau))^2 \right\} du.
\]

(B1)

The notation \( \sigma_z \) represents the partial derivative of \( \sigma \) with respect to its second argument. Given the expression for \( \sigma \) in equation (2), we can write,

\[
\sigma_z(u, t, r(u), f(u, \tau)) = -\lambda \sigma(u, t, r(u), f(u, \tau)).
\]

(B2)

Thus, the right hand side of equation (B1) becomes,

\[
\int_0^t \left\{ -\lambda \sigma(u, t, r(u), f(u, \tau)) \int_u^T \sigma(u, s, r(u), f(u, \tau)) ds + \sigma(u, t, r(u), f(u, \tau))^2 \right\} du.
\]

(B3)
Similarly for the stochastic term,

\[ d \int_0^t \sigma(u, t, r(u), f(u, \tau)) d\tilde{W}(u) \]

\[ = \left[ \sigma^2(u, t, r(u), f(u, \tau)) - \lambda \sigma(u, t, r(u), f(u, \tau)) \right] dt + \sigma(t, t, r(t), f(t, \tau)) d\tilde{W}(t) \]

\[ = \left[ -\lambda \int_0^t \sigma(u, t, r(u), f(u, \tau)) d\tilde{W}(u) \right] dt + \sigma(t, t, r(t), f(t, \tau)) d\tilde{W}(t). \]  \hspace{1cm} (B4)

Using results (B3) and (B4) we readily obtain the stochastic differential equation (11).

**Appendix C (Derivation of equation 15)**

Taking in turn differentials of (11), (12), (13), making use of (B1) and (B3) we obtain the stochastic differential equations for \( \psi \) viz

\[ d\psi(t) = \left[ \sigma(t, t, r(t), f(t, \tau))^2 + 2\int_0^t \sigma(u, t, r(u), f(u, \tau)) \cdot \sigma_2(u, t, r(u), f(u, \tau)) du \right] dt \]

\[ = \left[ \sigma(t, t, r(t), f(t, \tau))^2 - 2\lambda \int_0^t \sigma(u, t, r(u), f(u, \tau))^2 du \right] dt \]  \hspace{1cm} (C1)

\[ = \left[ \sigma(t, t, r(t), f(t, \tau))^2 - 2\lambda \psi(t) \right] dt. \]

**Appendix D (Proof of Proposition 1)**

Recall that \( r(t) \) satisfies the stochastic integral equation (9) and \( f(t, \tau) \), the stochastic integral equation (8) with \( T \) set equal to \( \tau \).

We have the forward rate volatility function

\[ \sigma(u, t, r(u), f(u, \tau)) = e^{-\lambda(t-u)} g[r(u), f(u, \tau)] \]  \hspace{1cm} (D1)

and set
\[
\sigma^*(u, t, r(u), f(u, \tau)) = \sigma(u, t, r(u), f(u, \tau)) \int_u^t \sigma(u, s, r(u), f(u, \tau)) ds
\]
\[
= e^{-\lambda (t-u)} g[r(u), f(u, \tau)] \int_u^t e^{-\lambda (t-u)} g[r(u), f(u, \tau)] ds
\]
\[
= g[r(u), f(u, \tau)]^2 e^{-\lambda (t-u)} \left[ 1 - e^{-\frac{\lambda (t-u)}{\lambda}} \right]
\]  \hspace{1cm} (D2)

Note that the second integral term in equation (9) can be written

\[
\int_0^t \sigma^*(u, t, r(u), f(u, \tau)) du
\]
\[
= \int_0^t e^{-\lambda t} g[r(u), f(u, \tau)]^2 e^{-\lambda u} \left[ 1 - e^{-\frac{\lambda (t-u)}{\lambda}} \right] du
\]
\[
= e^{-\lambda t} \int_0^t g[r(u), f(u, \tau)]^2 e^{-\lambda u} du - \frac{e^{-\lambda t}}{\lambda} \int_0^t g[r(u), f(u, \tau)]^2 e^{-2\lambda u} du
\]
\[
= e^{-\lambda t} I(t; \lambda) - \frac{e^{-2\lambda t}}{\lambda} I(t; 2\lambda). \hspace{1cm} (D3)
\]

Similarly the second integral term in (8) can be written

\[
\int_0^t \sigma^*(u, \tau, r(u), f(u, \tau)) du
\]
\[
= \int_0^t e^{-\lambda \tau} g[r(u), f(u, \tau)]^2 e^{-\lambda u} \left[ 1 - e^{-\frac{\lambda (\tau-u)}{\lambda}} \right] du
\]
\[
= e^{-\lambda \tau} \frac{\lambda}{\lambda} I(t; \lambda) - \frac{e^{-2\lambda \tau}}{\lambda} \frac{\lambda}{\lambda} I(t; 2\lambda). \hspace{1cm} (D4)
\]

Next note that the second integral in equation (9) may be written as

\[
\int_0^t \sigma(u, t, r(u), f(u, \tau)) d\bar{W}(u) = \int_0^t e^{-\lambda (t-u)} g[r(u), f(u, \tau)] d\bar{W}(u)
\]
\[
= e^{-\lambda t} \int_0^t e^{-\lambda u} g[r(u), f(u, \tau)] d\bar{W}(u)
\]
\[
= e^{-\lambda t} J(t, \lambda). \hspace{1cm} (D5)
\]

The second integral term in equation (8) may be similarly treated, so that
\[ \int_0^t \sigma(u, \tau, r(u), f(u, \tau)) \, dW(u) = \int_0^t e^{-\lambda(t-u)} g[r(u), f(u, \tau)] \, dW(u) \]
\[ = e^{-\lambda t} \left[ \int_0^t e^{iu} g[r(u), f(u, \tau)] \, dW(u) \right] \]
\[ = e^{-\lambda t} J(t, \lambda). \]  
\[ \text{(D6)} \]

We may thus write the stochastic integral equations for \( r(t), f(t, \tau) \) in terms of the integrals \( I(t; \lambda), I(t; 2\lambda) \) and \( J(t; \lambda) \) as

\[ r(t) = f(0, t) + \frac{e^{-\lambda t}}{\lambda} I(t; \lambda) - \frac{e^{-2\lambda t}}{\lambda} I(t; 2\lambda) + e^{-\lambda t} J(t; \lambda), \]  
\[ \text{(D7)} \]

\[ f(t, \tau) = f(0, \tau) + \frac{e^{-\lambda t}}{\lambda} I(t; \lambda) - \frac{e^{-2\lambda t}}{\lambda} I(t; 2\lambda) + e^{-\lambda t} J(t; \lambda). \]  
\[ \text{(D8)} \]

We note that equations (D7) and (D8) can be re-expressed as

\[ r(t) - f(0, t) + \frac{e^{-2\lambda t}}{\lambda} I(t; 2\lambda) = e^{-\lambda t} \left[ \frac{I(t; \lambda)}{\lambda} + J(t; \lambda) \right] \]  
\[ \text{(D9)} \]

\[ f(t, \tau) - f(0, \tau) + \frac{e^{-2\lambda t}}{\lambda} I(t; 2\lambda) = e^{-\lambda t} \left[ \frac{I(t; \lambda)}{\lambda} + J(t; \lambda) \right]. \]  
\[ \text{(D10)} \]

We may combine (D9) and (D10) to express \( I(t; 2\lambda) \) as a function of \( r(t) \) and \( f(t, \tau) \) i.e.

\[ I(t; 2\lambda) = \frac{\lambda e^{\lambda t}}{\lambda e^{\lambda t} - \lambda e^{-\lambda t}} \left[ f(t, \tau) - f(0, \tau) \right] - \frac{\lambda e^{\lambda t}}{\lambda e^{\lambda t} - \lambda e^{-\lambda t}} \left[ r(t) - f(0, t) \right] \]  
\[ \text{(D11)} \]

Finally we note that

\[ \psi(t) = \int_0^t \sigma(u, t, r(u), f(u, \tau))^2 \, du \]
\[ = \int_0^t e^{-\lambda(t-u)} g[r(u), f(u, \tau)]^2 \, du \]
\[ = e^{-\lambda t} \left[ \int_0^t e^{2iu} g[r(u), f(u, \tau)]^2 \, du \right] \]
\[ = e^{-\lambda t} I(t; 2\lambda). \]  
\[ \text{(D12)} \]

Thus from (D11) and (D12) we finally have

\[ \psi(t) = \lambda \alpha(t, \tau) [r(t) - f(0, t)] - \lambda e^{-\lambda(t-\tau)} \alpha(t, \tau) [f(t, \tau) - f(0, t)]. \]  
\[ \text{(D13)} \]
where we set
\[ \alpha(t, \tau) \equiv e^{-\lambda t} / (e^{-\lambda t} - e^{-\lambda \tau}). \]  

**Appendix E**

It is readily verified that the manipulations that led to equation (D8) of appendix D are equally valid for \( \tau \) set to a general maturity \( T \). Thus equation (D11) holds for \( \tau \) set to \( T \) i.e.

\[
I(t; 2\lambda) = \frac{\lambda e^{\lambda T}}{e^{-\lambda t} - e^{-\lambda T}} [f(t, T) - f(0, T)] - \frac{\lambda e^{\lambda T}}{e^{-\lambda t} - e^{-\lambda T}} [r(t) - f(0, t)] \\
= e^{2\lambda t} \psi(t), \quad \text{(By use of D12).}
\]

Substituting the expression for \( \psi(t) \) from (D13) and solving for \( f(t, T) - f(0, T) \) we obtain the expression given in proposition 2.

**Appendix F (Details of the infinitesimal generator \( K \))**

We recall the following result from Oksendal (1992) (theorem 7.9) concerning the infinitesimal generator of an \( n \) dimensional Ito process.

Consider the \( n \)-dimensional Ito stochastic differential system

\[
dX_i = a_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t), \tag{E1}
\]

where \( W_1(t), ..., W_n(t) \) are independent Wiener processes. Let \( \sigma \) denote the matrix whose elements are the \( \sigma_{ij} \) and define the matrix \( S \) as

\[
S = \sigma \sigma^T. \tag{E2}
\]

The infinitesimal generator \( K \) of the process \( X \) is given by

\[
K = \sum_{i,j} a_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} S_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{E3}
\]

So that the Kolmogorov backward equation for the transition probability density function of the process generating \( X \) is given by

\[
\frac{\partial \pi}{\partial t} + K \pi = 0. \tag{E4}
\]

In our application we set
\[ X_1 \equiv f(t, \tau), \]
\[ a_1 \equiv \sigma(t, \tau, r(t), f(t, \tau))^2 \frac{[e^{\lambda(t-\tau)} - 1]}{\lambda}, \]
\[ \sigma_{11} \equiv \sigma_1 \equiv \sigma(t, \tau, r(t), f(t, \tau)), \]
\[ X_2 \equiv r(t), \]
\[ a_2 \equiv f_2(0, t) - \lambda f(0, t) + \psi(t) - \lambda r(t), \]
\[ \sigma_{21} \equiv \sigma_r \equiv \sigma(t, t, r(t), f(t, \tau)). \]

Thus the matrix $S$ assumes the form

\[ S = \begin{pmatrix} \sigma_i^2 & \sigma_i \sigma_r \\ \sigma_i \sigma_r & \sigma_r^2 \end{pmatrix}. \] (E6)

Using the foregoing expression for $S$ the expression for $K$ in equation (27) is readily derived.

**Appendix G (Proof of Proposition 3)**

Using the relationship

\[ P(t, T) = \exp \left( - \int_t^T f(t, s) ds \right) \]

and equation (8) for the forward rate $f(t, s)$ we obtain for the bond price the expression

\[ P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left[ - \left( \int_t^T \int_s^T \sigma^*(s, u, t) du ds + \int_t^T \int_s^T \sigma(s, u, t) d\tilde{W}(s) du \right) \right], \]

where

\[ \sigma(s, T, t) = e^{-\lambda(t-s)} \sigma(r(s), f(s, \tau)) \]
\[ \sigma^*(s, T, t) = \sigma(s, T, t) \int_s^T \sigma(s, u, t) du \]
\[ = \sigma(r(s), f(s, \tau))^2 e^{-\lambda(t-s)} \int_s^T e^{-\lambda(s-u)} du \]

Set
\[
I = \int_0^T \int_0^T \sigma^*(s, u_\tau) \, ds \, du + \int_T^T \int_0^T \sigma(s, u_\tau) \, d\tilde{W}(s) \, du
= I_1 + I_2
= \int_0^T \int_0^T \sigma^*(s, u_\tau) \, ds \, du + \int_0^T \int_0^T \sigma(s, u_\tau) \, d\tilde{W}(s),
\]

where we have interchanged the order of integration to obtain the last equality. Next note that

\[
\int_0^T \sigma^*(s, u_\tau) \, du = \sigma(r(s), f(s, \tau)) \int_0^T e^{-\lambda(u-t)} \int_0^u e^{-\lambda(y-t)} \sigma(r(s), f(s, \tau)) \, dy \, du
= \sigma(r(s), f(s, \tau)) \int_0^T e^{-\lambda(u-t)} \left( \int_0^u e^{-\lambda(y-t)} \sigma(r(s), f(s, \tau)) \, dy \right) \, du
+ \int_0^u e^{-\lambda(y-t)} \sigma(r(s), f(s, \tau)) \, dy \right) \, du
= \sigma^2(r(s), f(s, \tau)) \int_0^T e^{-\lambda(u-t)} \int_0^u e^{-\lambda(y-t)} \, dy \, du
+ \sigma^2(r(s), f(s, \tau)) \int_0^T e^{-\lambda(u-t)} \int_0^u e^{-\lambda(y-t)} \, dy \, du
\]

\[
= \sigma^2(r(s), f(s, \tau)) \int_0^T \left( \int_0^u e^{-\lambda(y-t)} \, dy \right) \, du + \sigma^2(r(s), f(s, \tau)) \int_0^T \left( \int_0^u e^{-\lambda(y-t)} \, dy \right) \, du
\]

\[
= \sigma^2(s, t_\tau) \beta(t, T) + \sigma^2(s, t_\tau) \alpha(t, T)
\]

where

\[
\beta(t, T) = \int_0^T e^{-\lambda(u-t)} \, du = \frac{1}{\lambda} \left[ 1 - e^{-\lambda(T-t)} \right]
\]

\[
\alpha(t, T) = \int_0^T e^{-\lambda(u-t)} \int_0^u e^{-\lambda(y-t)} \, dy \, du = \frac{1}{2} \beta(t, T)^2
\]

i.e. we have shown that

\[
\int_0^T \sigma^*(s, u_\tau) \, du = \beta(t, T) \sigma^*(s, t_\tau) + \frac{1}{2} \beta(t, T)^2 \sigma^2(s, t_\tau).
\]

Next consider

\[
\int_0^T \sigma(s, u_\tau) \, du = \int_0^T e^{-\lambda(u-t)} \sigma(r(s), f(s, \tau)) \, du
= \sigma(r(s), f(s, \tau)) e^{-\lambda(t-t)} \int_0^T e^{-\lambda(u-t)} \, du
\]

i.e. we have shown that

\[
\int_0^T \sigma(s, u_\tau) \, du = \sigma(s, t_\tau) \beta(t, T).
\]
Returning to the expressions for $I_1$, $I_2$ we can now write

$$I_1 = \int_0^T \left\{ \beta(t, T) \sigma^*(s, t, \cdot) + \frac{1}{2} \beta(t, T)^2 \sigma^2(s, t, \cdot) \right\} ds$$

and

$$I_2 = \int_0^T \beta(t, T) \sigma(s, t, \cdot) d\tilde{W}(s)$$

so that

$$I = \frac{1}{2} \beta^2(t, T) \int_0^T \sigma^2(s, t, \cdot) ds + \beta(t, T) \left\{ \int_0^T \sigma^*(s, t, \cdot) ds + \int_0^T \sigma(s, t, \cdot) d\tilde{W}(s) \right\}.$$

However we note from equation (6), for the instantaneous spot rate $r(t)$, that

$$\int_0^T \sigma^*(s, t, \cdot) ds + \int_0^T \sigma(s, t, \cdot) d\tilde{W}(s) = r(t) - f(0, t).$$

Hence

$$I = \frac{1}{2} \beta^2(t, T) \int_0^T \sigma^2(s, t, \cdot) ds + \beta(t, T) \left[ r(t) - f(0, t) \right].$$

Recalling the definition of the subsidiary stochastic variable $\psi(t)$ we can finally write

$$I = \frac{1}{2} \beta^2(t, T) \psi(t) + \beta(t, T) \left[ r(t) - f(0, t) \right].$$

Hence the expression for the bond price may be written as in proposition 3.
References


Chiarella, C. And Kwon O.K. (1999a), "Formulation of Popular Interest Rate Models under the HJM Framework", Research Paper 13, Quantitative Finance Research Group, School of Finance and Economics, UTS.


