Modelling the Expected Value of the Classical Rescaled Adjusted Range for Long-Term Dependent Series

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Abstract

Hurst exponent estimates for long-term dependent series are well known to be biased with respect to their true value. Consequently, results based on the estimation of the classical rescaled adjusted range are conditional upon the choice of an appropriate benchmark, or expected value. The objective of this paper is to propose a framework for the estimation of the expected rescaled adjusted range for a general class of long-term dependent time-series.
1. **Introduction**

Traditional models of the dynamic behaviour of financial and economic asset yields are derived from the assumptions that successive price changes are statistically independent and conform to a Gaussian distribution. These assumptions have been used as the basis for the development of linear asset pricing models. However, recent theoretical and empirical literature has sought to develop various non-linear models as an alternative description of asset price dynamics. One type of deterministic non-linear model, which has been extensively used outside the mainstream literature is the classical rescaled adjusted range statistic.

Proposed by Hurst (1951) the classical rescaled adjusted range statistic \((R^*/\sigma_n)\) can be used to determine the direction and magnitude of dependence in the increments of a time-series. For a sample of returns \(X_1, \ldots, X_n\) the statistic is calculated as

\[
\frac{R^*/\sigma_n}{(l/\sigma_n)} = \left(\frac{1}{\sigma_n}\right) \left[ \max \sum_{k}^{n} (X_{j} - M_{n}) - \min \sum_{k}^{n} (X_{j} - M_{n}) \right]
\]

(1)

where \(M_n\) is the sample mean \((1/n)\sum X_j\) and \(\sigma_n\) is the standard deviation of the returns

\[
\sigma_n = \left[ \frac{1}{n} \sum_{j}^{n} (X_{j} - M_{n})^2 \right]^{0.5}
\]

(2)

For a series of length \(N\) the statistic's Hurst exponent \((H)\) may then be calculated by a simple OLS regression of the form
\[
\log \left( \frac{R^*}{\sigma} \right)_n = \alpha + \beta \log(n) + \varepsilon \tag{3}
\]

where given \( n \leq N \), the estimate of the value of the exponent is the regression \( \beta \) coefficient.

For a series of random Gaussian increments, the limiting value of the exponent is \( H = 0.5 \). However across a range of small to moderate lengths \( N \), the value of the exponent may be any \( H > 0.5 \). As such, the interpretation of the classical rescaled adjusted range is conditional on the use of an appropriate benchmark, or expected, value. Models of the expected rescaled adjusted range for random Gaussian series have been developed by Hurst (1951), Anis and Lloyd (1976), and Peters (1994). Using Monte Carlo simulation techniques, Ellis (1996) demonstrated that the Anis and Lloyd (1976) model best describes the asymptotic behaviour of \( (R^*/\sigma)_n \) for random Gaussian series.

Examining the characteristics of exponent \( H \) estimates for long-term dependent series, Mandelbrot and Wallis (1969) showed that mean estimates of \( (R^*/\sigma)_n \) overestimate their (true) underlying value for \( H < 0.7 \) and underestimate the underlying value for \( H > 0.7 \). However, since little is known about either the distribution of the exponent estimates, or their expected value \( E(R^*/\sigma)_n \), this result is only general.

Research in this paper proposes a model of the expected rescaled adjusted range for long-term dependent series with \( 0 < H < 1, H \neq 0.5 \). The approach adopted is to model the relative bias of the estimated value of the exponent \( H \) as a proxy for \( E(R^*/\sigma)_n \). The organisation of this paper is as follows. Section 2 discusses the application of existing models of \( E(R^*/\sigma)_n \) to modelling the expected rescaled adjusted range for long-term dependent series, and provides the motivation
for this study. Section 3 develops the empirical model of relative bias and discusses its application to modelling $E(R'/\sigma)_n$. Using simulated long-term dependent fractional Brownian motions (fBm's) and random Gaussian series, the power of the model is also discussed in this Section. Finally, Section 4 provides some concluding remarks.

2. Properties of the expected rescaled adjusted range.

Examining the properties of the classical rescaled adjusted range, for Gaussian series with $H = 0.5$, Wallis and Matalas (1970) suggested that such analysis should consider the mean, variance and relative bias of $(R'/\sigma)_n$ estimates. Relative bias was defined as the difference between the observed estimate of $(R'/\sigma)_n$ and its expected value $E(R'/\sigma)_n$, calculated using the Hurst (1951) model in Equation (4) where, for all values of the subseries length ($n$), the slope of the trendline is given by $n^{0.5}$:

$$E(R'/\sigma)_n = \left(\frac{n\pi}{2}\right)^{0.5}$$

(4)

For long-term dependent series with $0 < H < 1$; $H \neq 0.5$, Mandelbrot and Wallis (1969) proposed the relationship between $(R'/\sigma)_n$ and $n$ could be described by $(R'/\sigma)_n \sim n^H$. As before, the variable $H$ represents the underlying value of the Hurst exponent. Generalising the Hurst (1951) model by the replacement of 0.5 with $H$, it follows that

$$E(R'/\sigma)_n = (n\alpha)^H$$

(5)

for long-term dependent fBm's. For Gaussian series with $H = 0.5$, the value of the intercept term
(α) is π/2 and Equation (5) yields Equation (4).

Substituting the benchmark value π/2 for α, empirical tests results reveal that the generalised Hurst model is a poor descriptor of the expected value of (R'/σ)_n for long-term dependent series. The test methodology employed involved simulating 1000 series for each of the underlying values of the exponent H = 0.1, 0.3, 0.7 and 0.9. Comparing mean estimates of (R'/σ)_n from Equation (3) to their expected value in Equation (5), a large consistent divergence of (R'/σ)_n from E(R'/σ)_n may be seen. An implication of this finding is that the distribution of the underlying fBm series from which the estimates of (R'/σ)_n were derived, is non-normal. This follows from Ambrose, et al (1993) who proposed that under a normal Gaussian distribution, the intercept of the plot of log (R'/σ)_n versus log (n) would be such that α = π/2.

Empirical analysis using Equation (5) with various other consistent estimates of the intercept term and 0<H<1; H ≠ 0.5, similarly results in divergence between the plots of (R'/σ)_n and E(R'/σ)_n. Generally, given 0≤H<0.5 the bias (error) is divergent and positive and implies the value of the intercept in the Hurst (1951) model should be such that α<π/2. Alternatively given 0.5<H≤1, the divergent negative bias implies α>π/2. This general result implies the value of the intercept term may be indirectly related to the underlying value of the exponent H. The appropriate choice of intercept term using this methodology will be the value for which [(R'/σ)_n-E(R'/σ)_n]→0 as n→∞, such that (R'/σ)_n is convergent to its expected value.

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1. Long-term dependent fBm's were simulated using the Voss (1985) method of successive random addition. Details of the empirical findings, including the moments of the distributions of (R'/σ)_n estimates are available from the author by request.

2. Consistent estimates of the value of the intercept term employed herein included estimating both of (R'/σ)_n and E(R'/σ)_n with: zero intercept and an intercept equal to the estimated value of α for the regression log (R'/σ)_n = α+βlog(n)+ε.
Considering other functional forms of the expected rescaled adjusted range for Gaussian series, Peters (1994) suggested the Anis and Lloyd (1976) model of $E(R'/\sigma)_n$ could be reduced using Stirling's Function to

$$E(R'/\sigma)_n = (n\pi/2)^{0.5} \sum_{r=1}^{n-1} [(n-r)/r]^{0.5}$$  \hspace{1cm} (6)

Written in this form, the first term in Equation (6) is the reciprocal of the Hurst (1951) model in Equation (4). On the basis of the just described behaviour of the Hurst model for $0<H<1$; $H \neq 0.5$, it can be seen using the relation $\log (x) = -1 \log (1/x)$ (where $x = (n\pi/2)^{0.5} = 1/((n\pi/2)^{0.5})$), that estimates of $E(R'/\sigma)_n$ using both the Anis and Lloyd and Peters models should also be expected to diverge from the estimated $(R'/\sigma)_n$. Substituting $0<H<1$; $H \neq 0.5$ for the value $H = 0.5$ in the Anis and Lloyd and the Peters models, this theoretical result may also be empirically proven. The result implies that the Anis and Lloyd (1976) model and related Peters (1994) model are likewise inappropriate measures of $E(R'/\sigma)_n$ when the underlying series is long-term dependent with $0\leq H \leq 1$.

3. **Modelling bias as the proxy for expected value.**

By implication of the slow convergence of $(R'/\sigma)_n$ to its limiting value, estimates of relative bias can be seen to be conditional upon the functional form of $E(R'/\sigma)_n$ employed. In so far however that existing models of the expected rescaled adjusted range are only well defined for random Gaussian series, it follows that little is know about the exact asymptotic behaviour of $(R'/\sigma)_n$ estimates for long-term dependent series. Using Monte Carlo simulation techniques, relative bias can be modelled directly as a function of the underlying values of the exponent, rather than
E(R'/\sigma)_n$. For known values of the model parameters, the technique may also be adapted to estimate the expected rescaled adjusted range when the underlying series is long-term dependent.

Referring to Equation (7), the methodology employed for estimating relative bias (denoted $b_n$) is to express the bias in terms of the difference between the ratio of a consistent measure of the underlying value of the exponent $H$ to the logarithm of the subseries length ($n$), and the ratio of $\log (R'/\sigma)_n$ to $\log (n)$:

$$b_n = \left[ \frac{\hat{\alpha} + \hat{\beta} \log(n)}{\log(n)} - \frac{\log(E(R'/\sigma)_n)}{\log(n)} \right]$$

$$\left[ \frac{\hat{\alpha} + \hat{\beta} \log(n)}{\log(n)} - \frac{\log(n^H)}{\log(n)} \right] = \alpha_n + \beta_n \ln(\log(n)) + \varepsilon$$

The initial assumptions of the model are that the underlying value of $H$ is known, and that the regression of $\log (R'/\sigma)_n$ versus $\log (n)$ is for values of $n$ outside of the transient region. Consistent with Mandelbrot and Wallis (1968), the relevance of the latter assumption is to ensure the goodness of fit of the OLS estimate of $\log (R'/\sigma)_n$. The first assumption that the underlying value of $H$ is known, will be presently maintained to demonstrate the fit of the model, and can be relaxed. Integral to this specification of $b_n$, is that the dependent variable is expressed in ratio terms. As in the present case where the appropriate form of $E(R'/\sigma)_n$ is unknown for $0 \leq H \leq 1$; $H \neq 0.5$, the proxy for expected value in Equation (7) is taken to be $\log(n^H)$ and Equation (7) is rewritten as Equation (8).
The first term of the dependent variable in Equation (8) is the Mandelbrot and Wallis OLS regression equation for log \((R'/\sigma)_n\) in Equation (3). Expressed in terms of the ratio to log \((n)\), this term can be seen to be log-linear with respect to \(n\) in log-log space. Divided by log \((n)\), the second term of the dependent variable in Equation (8) then reduces to \(H\), and is linear across all values of \(n\). Given the linearity of \((R'/\sigma)_n\), it follows then that the difference \((b_n)\) is also log-linear. The convergence of the relative bias to zero over increasing values of \(n\) is shown in Figure 1 for various underlying values of the exponent \(H\). As shown in this Figure, the relative bias is greatest for values of the exponent near the limiting values \(H = 0\) and \(H = 1\), yet is asymmetric about the Gaussian value \(H = 0.5\).

**Figure 1. Relative Bias as a Function of the Exponent \(H\) and \(n\).**

![Graph showing relative bias as a function of exponent](image)

Using the methodology adopted in this study, the indicated relative bias is asymptotic even over finite values of \(n\). By comparison, the traditional methodology measures relative bias as the difference between two linear functions \((E(R'/\sigma)_n\) and \((R'/\sigma)_n\)\) which by definition are only 'trivially asymptotic'. Modelled as an asymptote, the proposed methodology may be seen to

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The term 'trivially asymptotic' is used here to describe the case where two linear functions converge slowly over large values of the independent variable (log \((n)\) in the case of the classical rescaled adjusted range). Such functions are typically characterised as having both similar intercept and slope coefficients.
allow accurate inferences about relative bias both when \( n \) is finite, and when the form of \( \text{E}(R'/\sigma)_n \) is unknown.

Parameter estimates of the value of \( \alpha_{b_n} \) and \( \beta_{b_n} \) from Equation (7), given various underlying values of the exponent \( 0 < H < 1 \) are provided in Table 1. The figures in parenthesis represent the standard error of the estimates. Based upon the mean of 1000 simulated \( (R'/\sigma)_n \) estimates each, the benchmark value of \( \text{E}(R'/\sigma)_n \) used herein is a linear curve whose slope is the underlying value of the exponent, and whose intercept is given to be \( \alpha_{\text{E}(R'/\sigma)_n} = 0 \).

<table>
<thead>
<tr>
<th>( H )</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept (( \alpha_{b_n} ))</td>
<td>0.41017 (0.00236)</td>
<td>0.22447 (0.00139)</td>
<td>0.03805 (0.00018)</td>
<td>-0.13961 (0.00117)</td>
<td>-0.32563 (0.00250)</td>
</tr>
<tr>
<td>Slope (( \beta_{b_n} ))</td>
<td>-0.13999 (0.00241)</td>
<td>-0.08253 (0.00142)</td>
<td>-0.01098 (0.00019)</td>
<td>0.06982 (0.00120)</td>
<td>0.14834 (0.00255)</td>
</tr>
<tr>
<td>MSE:</td>
<td>8.8E-06</td>
<td>3.0E-06</td>
<td>5.4E-08</td>
<td>2.2E-06</td>
<td>9.8E-06</td>
</tr>
</tbody>
</table>

The interpretation of the values of the intercept (\( \alpha_{b_n} \)) and slope (\( \beta_{b_n} \)) coefficients follows from usual OLS regression modelling. In the context of the current application, the sign of these coefficients provides information about \( (R'/\sigma)_n \) and \( b_n \). Specifically, for \( \alpha_{b_n} > 0 \) and \( \beta_{b_n} < 0 \) the estimated value of the intercept coefficient in the regression of \( \log (R'/\sigma)_n \) versus \( \log (n) \) is less than the hypothesised intercept (in the present case \( \alpha_{\text{E}(R'/\sigma)_n} = 0 \)), in which case \( b_n \) is positive and declining, and \( (R'/\sigma)_n \) exceeds \( \text{E}(R'/\sigma)_n \). Alternatively for \( \alpha_{b_n} < 0 \) and \( \beta_{b_n} > 0 \), \( b_n \) is negative and declining and \( \text{E}(R'/\sigma)_n \) exceeds \( (R'/\sigma)_n \).

Being linear and convergent, they must intersect at some value, and hence are not properly asymptotic.
Values of the slope coefficient ($\beta_{bn}$) in Table 1 can be seen to tend to zero in the neighbourhood of the underlying value of the exponent, $H = 0.7$ and indicate a reduction in relative bias towards zero in this region. This tendency, and the sign of the coefficients, is consistent with Mandelbrot and Wallis (1969) which showed mean estimates of ($R'/\sigma$)$_b$ were upwardly biased for $H<0.7$ and downwardly biased for $H>0.7$. The contribution of the result is to provide additional information about the magnitude of the bias, relative to the underlying value of the exponent $H$.

Examining the goodness of fit of Equation (7) in modelling $b_n$, the regression $R^2$ statistic, standard error of the coefficients, and Mean Squared Error (MSE) are each considered. For all underlying values of $0<H<1$ in Table 1, the regression $R^2$ is equal to 0.99033 and indicates a very large proportion of the variation in relative bias is explained by the model. Similarly, the standard error of the estimates of the intercept ($\alpha_{bn}$) and slope ($\beta_{bn}$) coefficients implies the values of these are all significantly different from zero. In so far that the both $R^2$ statistic and estimated standard errors may be biased by autocorrelation in the regression residuals, the Mean Squared Error of the model is also considered.⁴

Estimates of the MSE are based upon the mean of a further 1000 independent simulated values of ($R'/\sigma$)$_b$ each. Across all underlying values of the exponent, the forecast error indicated by the MSE is not significantly different to zero. Relative to the underlying value of the exponent, a direct linear relationship between $\beta_{bn}$ and $H$, and an indirect linear relationship between $\alpha_{bn}$ and

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⁴ The use of the MSE in the current application is justified on the basis that the size of the individual errors is approximately equal. In the case of some large individual errors, the Root Mean Squared Error (RMSE) would be recommended to be used (Gaynor and Kirkpatrick, 1994; 14).
H is apparent.\textsuperscript{5} For the known values of the parameters $\alpha_{bn}$ and $\beta_{bn}$ provided in Table 1, Equation (7) may be rearranged to solve for the proxy expected value of the exponent $H$:

$$\bar{E}(R^*/\sigma)_n = \hat{\alpha} + \hat{\beta}\log(n) = (\alpha_{bn} + \beta_{bn}\ln(\log(n)) + \varepsilon)\log(n) + \log(n^H)$$ \hspace{1cm} (9)

where the estimate of the $\beta$ coefficient is the proxy expected value. The significance of this empirical model of $E(R^*/\sigma)_n$ is that the parameter values may be easily determined using Monte Carlo techniques for other forms of long-term dependent processes than fBm.

By substitution of Equation (5) for $n^H$ in Equation (7), Table 2 presents estimates of the intercept coefficient for the generalised Hurst expected value model for $0 \leq H \leq 1$.\textsuperscript{6} In the case where the underlying series is assumed to be Gaussian with $H = 0.5$, the value of the intercept coefficient in Equation (5) is confirmed to be $\alpha_{E(R^*/\sigma)_n} = \pi/2$.

Consistent with the hypothesised behaviour of the intercept term in Section 2, it can be seen for $H < 0.5$ that $\alpha_{E(R^*/\sigma)_n} > \pi/2$, and for $H > 0.5$ that $\alpha_{E(R^*/\sigma)_n} < \pi/2$. Examining the relationship between the estimated value of $\alpha_{E(R^*/\sigma)_n}$ in Table 2 and the underlying value of the exponent $H$, it can be shown that this may be expressed as a power rule of the form $\log(\alpha_{E(R^*/\sigma)_n}^H) = \alpha + \beta(H) + \varepsilon$.

\textsuperscript{5} Modelled as a function of the underlying value of the exponent $H$, these relations are approximately $\alpha_{bn} = 0.50041 - 0.91784 (H)$ and $\beta_{bn} = -0.18532 + 0.36451 (H)$. Illustrating the goodness of fit of the relations, the MSE of these models is 1.1E-20 and 9.5E-21 respectively.

\textsuperscript{6} So as to avoid confusion with the relative bias intercept term ($\alpha_{bn}$), the intercept coefficient corresponding to the Hurst model in Table 2 is denoted as $\alpha_{E(R^*/\sigma)_n}$. In Equation (5) this was previously denoted simply as $\alpha$. 

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Table 2. Parameter Estimates for $b_n$ using Hurst (1951) Benchmark

<table>
<thead>
<tr>
<th>$H$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept ($\alpha_{E(R' / \sigma)_n}$)</td>
<td>8.2E+10\pi</td>
<td>15\pi</td>
<td>$\pi/2$</td>
<td>$\pi/6$</td>
<td>$\pi/13$</td>
</tr>
<tr>
<td>Intercept ($\alpha_{\theta}$)</td>
<td>-0.39638 (0.00463)</td>
<td>-0.16436 (0.00198)</td>
<td>-0.11387 (0.00110)</td>
<td>0.01275 (0.00014)</td>
<td>0.10437 (0.00123)</td>
</tr>
<tr>
<td>Slope ($\beta_{\theta}$)</td>
<td>0.27518 (0.00473)</td>
<td>0.11763 (0.02029)</td>
<td>0.06722 (0.00116)</td>
<td>-0.00862 (0.00015)</td>
<td>-0.07301 (0.00126)</td>
</tr>
<tr>
<td>MSE:</td>
<td>3.4E-05</td>
<td>6.2E-06</td>
<td>2.0E-06</td>
<td>3.3E-08</td>
<td>2.4E-06</td>
</tr>
</tbody>
</table>

Estimates of the value of $\alpha_{\theta}$ and $\beta_{\theta}$ in Equation (7) using the generalised Hurst model as the measure of expected value (and their respective standard errors) are presented in Table 2. When compared to the Hurst benchmark, the value of intercept coefficient in Equation (7) and exponent $H$ may be seen to be directly related, and the slope coefficient and $H$ indirectly related.

Consistent with results described in Table 1, the value of the slope coefficient ($\beta_{\theta}$) tends to zero as the underlying value of the exponent approaches $H = 0.7$, indicating a reduction in the relative bias of $(R' / \sigma)_n$ estimates. The sign of the intercept term ($\alpha_{\theta}$) in this model provides information about the direction from which $(R' / \sigma)_n$ tends to the expected value. For values of $\alpha_{\theta} < 0$, $(R' / \sigma)_n$ will tend to $E(R' / \sigma)_n$ from below, in which case $(R' / \sigma)_n$ is upwardly biased with respect to $H$. For $\alpha_{\theta} > 0$, the estimate of $(R' / \sigma)_n$ will converge to $E(R' / \sigma)_n$ from above, and will thus underestimate the underlying value of $H$. As in the case where the functional from of $E(R' / \sigma)_n$ is unknown, estimates of the Mean Squared Error (MSE) in Table 2 illustrate the goodness of fit of the model for the Hurst benchmark.
4. **Concluding remarks.**

Conclusions derived from the classical rescaled adjusted range are conditional upon the choice of an appropriate benchmark against which observed values of \((R'/\sigma)_n\) can be compared. Various models of the expected value of the classical rescaled adjusted range are attributed to Hurst (1951), Anis and Lloyd (1976) and Peters (1994). However, these models are only well defined for random Gaussian series and do not accurately describe the asymptotic behaviour of \((R'/\sigma)_n\) estimates derived from long-term dependent fBm's.

Relative bias has been defined as the difference between the observed and expected values of \((R'/\sigma)_n\) estimates. The determination of the relative bias of the Hurst exponent, is however conditional upon the choice of an appropriate benchmark model of the expected behaviour of \((R'/\sigma)_n\).

In the case where the correct functional model of expected value is unknown, the technique developed in this study was to model bias directly as a function of the underlying value of the exponent \((H)\) and the subseries length \((n)\). The benefit of the proposed methodology approach was that the relative bias of \((R'/\sigma)_n\) could in fact be modelled independent of the underlying functional form of \(E(R'/\sigma)_n\). By substitution of the Hurst (1951) model of \(E(R'/\sigma)_n\), the general model of relative bias has also been adapted to include specific traditional models of expected value.

Providing a framework for the empirical estimation of \(E(R'/\sigma)_n\) for alternative models of long-term dependence, an important contribution of this research has been to demonstrate a consistent and systematic relationship between the level of relative bias and the underlying
value of the exponent H. The implication of this result is that the classical rescaled adjusted range statistic may be effectively used with long-term dependent processes with $0 \leq H \leq 1$, once consideration for the bias has been given.
References


