Towards Applied Disequilibrium Growth Theory: V Housing Investment Cycles, Private Debt Accumulation and Deflation

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Abstract

In this paper we reconsider a general disequilibrium model of an applied orientation, exhibiting a detailed modelling of the private housing sector, which we have developed in a series of working papers starting from the Murphy model for the Australian economy. This modelling approach is complete with respect to budget equations and stock-flow interactions and can be reduced to a somewhat simplified 18D core model, the dynamics of which was intensively studied in the earlier work. In the present paper we modify this type of model towards the explicit consideration of debtor and creditor households which extends the dynamics of the core model by 1 to 19D by the addition of the dynamics of the debt to capital ratio of indebted households. Various subdynamics of these 19D dynamics are investigated theoretically and illustrated numerically. The basic findings are that there is convergence to the balanced growth path of the model for sluggish disequilibrium adjustment processes, that persistent investment cycles in the housing sector can be generated for certain higher adjustment speeds by way of Hopf-bifurcations in particular. Furthermore processes of debt deflation may trigger monotonic depressions that get more and more severe when the real debt of debtor households is systematically increased by deflationary spirals in the goods manufacturing sector in particular.

Keywords: Structural macroeconometric models, housing, stability, investment cycles, debt accumulation, deflation
JEL Classification: E12, E32.
1 Introduction

A proper modelling of the housing sector in a structural macroeconomic model needs to consider housing investment, the purchase of houses or housing services and the evolution of the prices charged for houses or housing services. The main focus in the applied literature on this issue has often been the subsector of office space, but of course the sector of privately owned houses or private rental is also a very large and important sector of the macroeconomy. What is particularly interesting in this type of literature from the macrodynamic point of view is that there are concepts and issues in the literature on the housing sector that are closely related to important topics of standard macrodynamic theorizing. There is the concept of the natural vacancy rate, see for example Gabriel and Nothaft (2000) or of a NAIRU in the housing sector, see Hendershott, MacGregor and Tse (2002), the concept of overbuilt markets, see Hendershott (1996) and of persistent cycles in the housing sector that in our view bear close resemblance to what is happening in the unemployment-inflation dynamics in the interaction of the labor market with the market for goods and the wage price spiral.

Due to the size of the housing sector it is therefore of great interest from the macrodynamic point of view to not only study this sector in its own interaction of space demand and supply, rental and housing prices and the rate of return they imply and finally the investment behavior in this sector, but to also consider its interaction with the rest of the macroeconomy where therefore then two real cycle generators may be at work leading to coupled oscillators or business cycles. In the present paper we want to lay foundations for such an investigation and to show that models of such type can even be handled from the theoretical perspective. Ultimately however numerical and empirical investigations will be needed, which however is beyond the scope of this paper.

In this paper we apply the general framework introduced and discussed in detail in Chiarella and Flaschel (1999a,b, 2000a) and Chiarella, Flaschel and Zhu (2003)\(^1\) to the special issue of housing investment cycles, the supply and demand for dwelling services and the price dynamics this implies in the housing sector. Rents in this sector then in turn determine rates of return for housing investment and interact with this investment in the generation of damped, persistent or even explosive cycles thus generated in the housing sector of the economy. The necessary ingredients for this analysis have by and large already been supplied in the works just quoted, but are here modified to a certain extent in order to allow a more specific analysis of the topics mentioned above. The following section will present in this regard the details we need for the following analysis of the dynamics originating in the housing sector and interacting with the general business cycle of the model. However we will not repeat the general framework of the papers Chiarella and Flaschel (1999a,b, 2000a) and Chiarella, Flaschel and Zhu (2003) in all of it details. The reader is therefore referred to these works for the full models on the extensive form level, the intensive form level and the many subdynamics to which this model type can give rise as well as investigations of their interaction in the integrated 18D core dynamics of this approach. This overall framework has been motivated by an attempt to understand the basic dynamic feedback mechanisms, and their interaction, of large scale macroeconomic models such as that of Powell and Murphy (1997).

The new focus that the current paper brings is on debt relationships in the household sector, composed of indebted worker households and pure asset holders as creditors. Worker households purchase houses as durable consumption goods (in addition to housing services by part of them) through credit from the asset holders and are thus now characterized by negative

\(^1\)See also Chiarella, Flaschel, Groh and Semmler (2000, part III) in this regard.
bond holdings in the place of positive bond holdings in the quoted earlier works. They indeed have – in the aggregate – a marginal propensity to consume out of their disposable income that is larger than 1 (when consumption of non-durables and durables is taken together) and they finance the excess of consumption over their disposable income by new debt and thus credit from the asset holding part of the household sector. In the steady state we will have a constant debt to capital ratio and thus debt of workers growing at a constant rate over time. Our interest however is to study endogenous fluctuations around this steady growth path and to investigate in addition the possibility that price and wage dynamics may be such that processes of debt deflation are generated, here not as in Chiarella, Flaschel and Semmler (2001a,b) with respect to firms and their indebtedness due to past investment behavior, but rather within the household sector and the debtor-creditor relationship.\(^2\)

In section 2 we present the new components of the model on the extensive form level, in addition to the details from our earlier work that concern the housing sector. Section 3 then derives the laws of motion of the general 19D dynamics, which – with special focus on the housing sector – are simplified thereafter to give rise to the core 9D dynamics to be investigated in the remainder of the paper. Section 4 considers 2D to 5D subcases of the integrated 6D real subdynamics of the 9D dynamics and formulates and proves a number of propositions on these subdynamics. The full 9D dynamics are investigated briefly from the numerical perspective at the end of section 4. Section 5 concludes.

2 Debt relationships in the household sector

We reformulate in this section the general model of disequilibrium growth introduced and investigated in Chiarella and Flaschel (1999a,b, 2000a) with respect to assets supplied and demanded by the two types of agents, workers and asset holders, assumed there to make up the household sector and with respect to the use that is made of these assets in that earlier work.\(^3\) We will assume that worker households as a total rent housing services and buy new houses (to some extent), and that they finance the resulting excess of their consumption over their disposable income via credits (bonds of the fixed price / variable interest variety of the model of Chiarella and Flaschel (1999a,b, 2000a)). These bonds are supplied by the other type of household of the model, the pure asset holders. Firms produce for domestic purposes a unique good that can be used as a consumption good proper by the two types of households considered, as a business fixed investment good, as an investment good providing housing services to the workers by asset holders, for the purpose of government consumption and now also for representing the direct consumption of houses by both asset holders and workers. These alterations of the original 18D core dynamics considered in Chiarella and Flaschel (2000a) will increase the dimension of these dynamics by one, since debt accumulation of workers will now feed back into the rest of the dynamics due to their consumption habits. We will represent the resulting dynamical system in compact form in the next section.

\(^2\)We add here that the analysis in Chiarella, Flaschel and Semmler (2001a,b) may indeed be applicable to the sector of office space, while the present paper focuses on private space.

\(^3\)If households are to be considered as heterogeneous in a macromodel then this should be the fundamental distinction in a model with labor supply and asset markets and with only two household types. Such polar types of households, which still appear in a very stylized way in the model presented here, are in our view much more relevant for macroeconomic model building than the distinction between workers and pensioners made in the OLG type models.
2.1 Worker househoolds

We consider the behavioral equations of worker households first, but only to the extent they are changed by the existence of a debtor-creditor relationship between our two types of agents in the household sector. In order to derive the new characteristics of this module of the model let us first present these equations in the form that were used in the original approach by Chiarella and Flaschel (1999a):

Households (Workers, original formulation):

\[ Y_{w}^{Dn} = (1 - \tau_w)[wL^d + w^u(L - L^u) + w^r\alpha_1 L_2] + (1 - \tau_c) rB_w = Y_{w1}^{Dn} + (1 - \tau_c)rB_w, \]
\[ p_yC_g^w = c_1 Y_{w1}^{Dn}, \]
\[ p_hC_h^w = c_2 Y_{w1}^{Dn}, \]
\[ S_w^n = Y_{w}^{Dn} - p_yC_g^w - p_hC_h^w = \dot{B}_w. \]

In these equations, the expression \( Y_{w}^{Dn} \) denotes total nominal disposable income of workers after taxes at the rate \( \tau_w \) as far as their labor income \( wL^d + w^u(L - L^u) + w^r\alpha_1 L_2 \) is concerned,\(^4\) and after taxes at the rate \( \tau_c \) with respect to their interest income \( rB_w \), on the stock \( B_w \) of short-term bonds accumulated by workers. We did also assume in the original approach of Chiarella and Flaschel (1999a) that workers save and thus hold and accumulate bonds in the amounts \( B_w \) and \( \dot{B}_w \), respectively, but there they reinvested all of their interest income into bond accumulation which thus did not feed back into the income term \( Y_{w}^{Dn} \) that determined their nominal consumption of goods \( p_yC_g^w \) and of housing services \( p_hC_h^w \) (with marginal propensities to consume \( c_1 + c_2 < 1 \)). These assumptions helped to simplify considerably the dynamics of that earlier study since the bond accumulation of workers did not influence aggregate demand and goods market behavior in this case. Nominal savings \( S_w^n \) of workers is invested into short-term bonds solely, since money is not a financial asset in the model of Chiarella and Flaschel (1999a,b, 2000a).

In the case of negative savings and thus debt accumulation (a negative \( B_w \) to be denoted by the positive expression \( D_w \) in the following), things are however not so easily disentangled, since debt is in fact made to increase the consumption of workers (to not only rent houses as was so far assumed, but also to buy to some extent houses as durable consumption goods, just as asset owners, thereby becoming debtors to asset holders just as the government. Interest payments must then appear in the income expression to be used for determining the consumption demand of workers since these payments reduce the possibility of workers to spend more than they earn.\(^5\) Assuming such a situation leads us to the following reformulation of the above representation of workers’ behavior, based on the augmented form of short-term loans, \( \dot{B}_c = \dot{D}_g + \dot{D}_w \), supplied by the asset holders to the government as well as to worker households.

Households (Workers, new formulation):

\[ Y_{w}^{Dn} = (1 - \tau_w)[wL^d + w^u(L - L^u) + w^r\alpha_1 L_2] - (1 - \tau_c) rD_w \]

\(^4\)Labor income here consists of wage income, unemployment benefits and pension payments, which are all subject to tax payments here at the uniform wage tax rate \( \tau_w \). Note however that the model would not be changed very much if differentiated wage tax rates are allowed for, an observation which also applies to the consumption propensities shown which at present are the same for employed, unemployed and retired workers.

\(^5\)Note here that interest payments are deducted before worker households decide on their consumption patterns. In the case where propensities to consume are applied to total wage income (after taxes) the dynamics of the capital to debt ratio to be considered later on do not feed back into the rest of the dynamics, which would then make the integrated dynamics of the type considered in Chiarella and Flaschel (1999a).
\begin{align}
Y_w^Dn &= (1 - \tau_c) rD_w \\
p_y C_g^w &= c_1 Y_w^Dn \\
p_h C_h^w &= c_2 Y_w^Dn \\
\dot{D}_w &= p_y C_g^w + p_h C_h^w - Y_w^Dn
\end{align}

We assume in this paper that \( c_1 + c_2 > 1 \) holds, i.e. worker households always consume more than they earn (after the deduction of interest payments).\(^6\) Assuming \( c_1 + c_2 > 1 \) for worker households amounts to assuming that there is no intertemporal budget constraint in the usual sense of the word for this type of household, just as for the government sector. In both cases we will have a given debt to capital ratio in the steady state meaning that part of the expenditure is always financed by issuing new debt which then grows (just as the stock of debt) with the given real growth rate of the world economy. Such an approach is admissible in a descriptively oriented disequilibrium growth model of monetary growth, in particular if it is understood that this model type (and its steady state solution) is to be applied to particular periods of the evolution of actual market economies. Assuming no debt of the government and of workers in the steady state by choosing the parameters of the model appropriately clearly is too limited an approach from the descriptive point of view.

We also note here that debt accumulation, and even more so debt deflation, is still of a fairly simple type here. Looking at the equations in the above module of the model clearly shows that workers' consumption demand depends negatively (due to their interest payments and their marginal propensities to consume being in sum larger than one) on their debt and thus on the debt to capital ratio \( d_w = D_w/(p_yK) \) to be considered later on. This means that aggregate demand depends negatively on this ratio \( d_w \) and will thus shrink when this ratio is increasing, which happens in particular when there is goods price deflation (taken in isolation). Such deflation therefore decreases aggregate demand, which via the Metzlerian goods market adjustment process leads to still lower economic activity and from there to further falling goods prices and so on. In this way a deflationary spiral may be established that drives the economy into more and more depressed situations. The resulting downward spiral in prices and wages and in economic activity depends however on the precise way wages, goods prices and rental prices are falling and what happens to other component of aggregate demand, i.e., we here have still ignored other feedback channels, in particular the ones that concern the effects of falling goods and rental prices.

One can interpret the above description of the behavior of worker households also in the following way. Assume that workers accumulate debt basically due to their purchase of houses, which can be made explicit if it is assumed that \( c_1 \) is split into \( c_1^D \) and \( c_1^h \), where the first parameter describes the propensity to consume consumption goods proper and where the second parameter denotes that portion of goods consumption that goes into purchase of houses. The equation that describes the evolution of the stock of houses \( K_h^w \) owned by workers is then given by:

\[
\dot{K}_h^w = c_1^h Y_w^Dn / p_y - \delta_h K_h^w \quad \text{or} \quad \dot{K}_h^w = c_1^h y_w^D / k_h^w - \delta_h - \hat{K},
\]

where \( \delta_h, y_w^D \) and \( k_h^w \) denote the depreciation rate of houses, the real disposable income of workers (after interest deduction) per unit of capital \( K \) and \( K_h^w / K \), respectively. The steady state value of the stock of houses owned by workers per unit of capital is thus given by

\[
k_h^w = c_1^h y_w^D / (\gamma + \delta_h),
\]

where \( \gamma \) denotes the steady state rate of growth of the economy. We shall show in section 4 that the steady state value of the debt to capital ratio \( d^w = D^w / (p_yK) \) is

\(^6\)Note that we have assumed here for reasons of symmetry that interest paid on debt leads to tax reduction at the rate \( \tau_c \), which however is a detail of the model which is of secondary importance.
given by \((c_1 + c_2 - 1)y_w^D/\gamma\). Assume now that \(D^w\) can be considered as the housing mortgage that can and will be bequeathed to the next generation if the side condition \(D^w \leq p_y K^w_h\) is fulfilled (since the mortgage is then less than the reproduction value of the stock of houses owned by worker households). In the steady state this side condition amounts to

\[
d_w = \frac{c_1 + c_2 - 1}{\gamma} y_w^D \leq \frac{c_1^h}{\gamma + \delta_h} y_w^D, \text{ i.e., } \frac{c_1 + c_2 - 1}{\gamma + \delta_h} \gamma \leq c_1^h,
\]

which thus gives a lower bound on the propensity to consume \(c_1^h\) such that the above side condition is fulfilled, at least along the steady state solution of the dynamics.

Note also that the dynamics of the model is based on a Keynesian determination of the short-run throughout so that demand is always satisfied in this model type. Situations where this is not the case are analyzed in detail in Chiarella, Flaschel, Groh and Semmler (2000) and do not lead to important changes in the behavior of the models there. Due to the two consumption functions just presented we have that Keynesian goods market features now depend on the stock of debt of workers (through the interest payments they imply). The debt to capital ratio of workers will therefore now influence the dynamics of the real part of the model in contrast to the situation considered in Chiarella and Flaschel (1999a,b, 2000a) where workers accumulated a positive stock of short term bonds.

### 2.2 Pure asset holder households

Next, we consider the other type of households of our model, the (pure) asset holders who are assumed to consume \(C_c\) (goods and houses as supplied by firms through their domestic production \(Y\)) at an amount that is growing exogenously at the rate \(\gamma\), which is thus in particular independent of their current nominal disposable income \(Y_c^Dn\). The consumption decision is thus not an important decision for pure asset holders. Their nominal income diminished by the nominal value of their consumption \(p_yC_c\) is then spent on the purchase of financial assets, three types of bonds (short-term, long-term domestic and foreign ones) and equities, as well as on investment in housing supply (for rent to part of the worker households). Note here that the one good view of the production of the domestic good entails consumption goods proper and houses so that asset holders buy houses for their consumption as well as for investment purposes.

#### Households (Asset-Holders):

\[
Y_{c}^{Dn} = (1 - \tau_c)[\rho^r p_y K + r B_c + B_1 + p_h C_h^w - p_y \delta_h K_h] + e(1 - \tau_c^*) B_2,
\]

\[
\dot{C}_c = \gamma,
\]

\[
S_c^a = Y_{c}^{Dn} - p_y C_c
\]

\[
= \dot{B}_c + \dot{B}_1/r^l + e \dot{B}_2/r^{l*} + p_y \dot{E} + p_y (I_h - \delta_h K_h), \quad \dot{B}_c = \dot{D}_g + \dot{D}_w,
\]

\[
C_h^w = K_h,
\]

\[
\rho_h = (p_y C_h^w - \delta_h p_y K_h)/(p_y K_h),
\]

\[
g_h = I_h/K_h
\]

\[
= \alpha_1^h((1 - \tau_c)\rho_h - \rho^r) + \alpha_2^h (r^l - r) + \alpha_3^h (\frac{C_h^w}{C_h} - \bar{U}_h) + \gamma + \delta_h,
\]

\[
\dot{p}_h = \beta_{ph}(\frac{C_h^w}{C_h} - \bar{U}_h) + \kappa_h \ddot{p}_y + (1 - \kappa_h) \pi^l,
\]

\[
K_h = I_h/K_h - \delta_h.
\]
This part of the model is basically the same as the one considered in Chiarella and Flaschel (1999a,b, 2000a), with the introduction of the exceptional case that asset holders in addition now lend to worker households. Equation (2) defines the disposable income of asset holders which consists of the dividend payments of firms (who distribute their whole expected profit to equity holders), interest on government bonds (short-term bonds and consols), \( r_B + B^*_t \), insofar they are held by domestic residents, rents for housing services net of depreciation, and interest payments on foreign bonds held by domestic households (after foreign taxation and expressed in domestic currency by means of the exchange rate \( e \)). Private savings of asset holders \( S^*_t \) thus also concerns short-term and long-term bonds (domestic and foreign ones with respect to the latter), equities and net housing investment. The supply of housing services \( C^*_t \) is assumed to be proportional to that part of the existing stock of houses \( K_t \) devoted to the supply of such services (here the factor of proportionality is set to unity for simplicity). We assume for simplicity that there is no resale market for houses as there are for the financial assets of the model, which however is a feature of the model that should be removed in further extensions. Note again that the production of houses is part of the production activities of firms and thus part of the homogeneous supply of the domestic (non-traded) output.

The demand for housing services \( C^*_t \) has already been defined in the preceding module. We assume that housing demand is always satisfied and we can guarantee this in general – up to certain extreme fluctuations in the demand for housing services – by assuming there is a given benchmark rate of capacity utilization \( U_t \) of the housing service supply beyond which there is additional pressure on the price \( p_t \) of housing services and also increased effort to invest into housing supply (which may be of such extent that rationing on the market for housing services is prevented). We have assumed in the workforce sector that their demand for housing services grows (apart from short term deviations) with trend rate \( \gamma \) (underlying the steady state of the model). This implies that housing services per household grow with trend rate \( \gamma - n \), where \( n \) is the natural rate of growth of the workforce. Therefore, over the growth horizon of the economy considered here, we have that worker households consume more and more housing services (measured by square meters per housing unit for example). Equation (10) describes the rate of gross investment into housing of asset holders, which depends on the profit rate \( \rho_t \) in the housing sector compared with the required rate of return, which is measured in reference to government consols by \( \rho^* = \tau_t - \pi_t \) (via Tobin’s \( q \) as the relative profitability measure). It depends furthermore on the interest spread \( \tau_t - \tau \) as a measure for the tightness of monetary policy and its perceived (or only believed) effects on the level of economic activity and employment, on the actual rate of capacity utilization of housing services in its deviation from the natural rate of occupancy (representing short-run demand pressure), \( \frac{C^*_t}{C^*_B} - U_t \), on the trend rate of growth \( \gamma \) and on the rate of depreciation \( \delta_t \) in the housing sector.

In equation (11) the rate of inflation of the rental price of housing services, \( \dot{p}_t \), depends as investment on the rate of capacity utilization in the housing sector (the demand pressure component) and on a weighted average formed by the actual rate of inflation of producer prices in the production of the domestic good and on the level of this inflation that is expected over the medium term as a medium-term average, the rate \( \pi_t \), whose the law of motion will be provided later on (this weighted average represents the cost-push component in this dynamical equation for the price of housing services). Finally in equation (12) actual gross investment

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7See for example Shilling, Sirmans and Corgel (1987) and Rosen and Smith (1983) for such NAIRU approaches to the market for housing services.

8Such a construction is needed for the discussion of steady states of the considered economy and of course is only applicable over certain periods of time in the evolution of market economies.
plans are always realized and thus determine the rate of growth of the housing stock by deducting the rate of depreciation from them.

Summing up we can state that the consumption decisions of asset owners are basically driven by exogenous habits\(^9\) that are independent of their income and wealth position and that their investment decision into housing service supply precedes the other (financial) asset accumulation decisions. These latter decisions are in the present framework governed by supply side forces based on the new issuing of bonds by the domestic government and of equities by firms. Furthermore, their choice of accumulation or decumulation of foreign long-term bonds is here determined as the residual to all these flows in or out of short-term and long-term domestic debt and the flow of new equities issued by firms and is thus determined as a last step in the savings decision of asset holders. The essential decisions in this module of the model are therefore the housing investment decision and the pricing rule for housing services. Due to the assumptions made on the consumption of asset holders we need not consider the asset accumulation of these agents explicitly in the dynamical investigations that follow.

2.3 Wage, price, and interest rate adjustment processes

Finally, we present the wage, price and interest rate dynamics of the model that are important for the integrated core 5D dynamics of the real part of the model to be investigated in this paper. This type of dynamics has started to receive growing attention in recent studies with an empirical orientation.\(^10\) We stress however that we do not yet pay attention to consumer price indices and the role of import prices in the formation of the money wage and the price level Phillips curves, respectively, see however Chiarella, Flaschel, Groh, Köper and Semmler (1999a,b) for such additions to the model. This module is the same as the one employed in Chiarella and Flaschel (1999a,b, 2000a), which in sum means that the basic change in this paper with respect to these earlier integrated models of monetary growth solely concerns the budget restriction and the consumption behavior of worker households.

\[
\begin{align*}
\ddot{w} &= \beta_w(V - \bar{V}) + \kappa_w(\dot{p}_y + \lambda) + (1 - \kappa_w)(\pi^t + \lambda), \quad V = L^d/L, \quad (13) \\
\ddot{p}_y &= \beta_p(U - \bar{U}) + \kappa_p(\dot{w} - \lambda) + (1 - \kappa_p)\pi^t, \quad U = Y/Y^r, \quad (14) \\
\dot{\pi} &= \beta_\pi(\alpha_\pi(\dot{p}_y - \pi) + (1 - \alpha_\pi)(\pi - \pi)), \quad (15) \\
\dot{r} &= -\beta_{r_1}(r - r^*_1) + \beta_{r_2}(\dot{p}_y - \bar{\pi}) + \beta_{r_3}(U - \bar{U}). \quad (16)
\end{align*}
\]

In equation (13) wage inflation \(\ddot{w}\) responds in the traditional Phillips curve manner to the state of the demand pressure in the labor market as measured by the deviations of the rate of employment \(V\) from its NAIRU-level \(\bar{V}\) and there is also the usual accelerator term of price inflation which is here measured as a weighted average of actual price inflation based on short-term perfect foresight (plus the actual rate \(\lambda\) of productivity growth) and expected medium-term price inflation (also augmented by the given rate of productivity growth). The law of motion (14) for goods prices \(p_y\) of the domestic commodity is formulated in a similar way, as a second type of Phillips curve. We use the demand pressure measure \(U - \bar{U}\), the deviation of actual capacity utilization of firms from its norm, as the demand pressure cause

\(^9\)Here in a way that allows for a fixed parameter representation in the intensive form of the model.

\(^10\)See Fair (2000) for a recent discussion of such Phillips curves and Laxton et al. (1997) with respect to the interest rate policy rule employed here.
of price inflation. The cost push term in the price inflation equation is given as a weighted average of current wage inflation and the one expected for the medium-run (both made less severe in their influence on price inflation by the existence of a positive growth rate of labor productivity).

In equation (15) expected medium-term inflation $\pi^t$ in turn is based on a weighted average of two expectations mechanisms, an adaptive one with weight $\alpha_r$ and a foreword looking one with weight $1 - \alpha_r$. Forward looking expectations are here simply based on the inflation target of the central bank $\bar{\pi}$, in the usual way of a regressive scheme of expectations revision. Inflationary expectations are thus following a weighted average of actual inflation and the target rate of the monetary authority. We assume $\bar{\pi} = 0$ in the following and thus will have no inflation in the steady state of the model. Furthermore we will also not consider the destabilizing role of inflationary expectations (the so-called Mundell-effect) in this paper and thus will set $\beta_{\pi t} = 0$ here for reasons of simplicity, in order to concentrate on destabilizing real debt and real wage adjustments.

We can see from the above description that only the inflation rate of the domestic good matters in the wage price module of our economy. Housing, i.e., the rental price of dwellings (and its rate of change $\bar{p}_n$) is thus ignored at present in this description of the wage price interaction. This simplifies the feedback structure of the model, but should give way to a domestic price index of the form $p = p^m + p^h$ and its rate of change in the wage equation in future reformulations of the model, see Chiarella, Flaschel, Groh, Köper and Semmler (2000) in this regard.

The interest rate adjustment rule, equation (16), of the monetary authority attempts to adjust the short-term interest rate $r$ towards the given rate of interest in the world economy, but also pays attention to the deviation of the actual rate of price inflation from the targeted one, implying rising $r$ if the actual rate is above the target (and vice versa). Finally, interest rates are more easily increased if there is demand pressure on the market for goods than in the opposite situation.

We do not go here into the other modules of the model as formulated in Chiarella and Flaschel (1999a,b), since they by and large will not matter very much in our subsequent investigation of housing investment cycles, consumer debt and wage deflation. These modules concern the sector of firms with its fixed proportions technology (including exports and imports) and an investment behavior that is similar to the one assumed for asset holders with respect to the housing stock, the government sector whose fiscal policies do not matter here (due to assumptions to be made in the next section), but which makes use of a Taylor type interest rate policy rule (as shown above), asset markets that adjust towards a general prevalence of interest rate parity conditions and Metzlerian adjustment of inventories and sales expectations of firms that generally do not correctly perceive aggregate demand on the market for domestic goods. These equations will be summarized in compact form on the level of intensive or state variables in the next section of the paper.

3 Intensive form derivation of a simplified 9D dynamics

In this section we present our modification of the 18D core model, which was investigated in Chiarella, Flaschel and Zhu (2003) from various numerical perspectives, in order to focus now on a detailed consideration of the possibility of housing cycles and the debt financing of the investment undertaken by workers into housing purchases as already contained, but not yet paid attention to, in this original approach to disequilibrium growth dynamics (where interest payments in the sector of worker household did not yet have an impact on their
consumption behavior). To simplify the model slightly we assume throughout the following that $C_e(0) = 0$ holds initially (and thus for all times) and thus neglect the consumption of asset holders altogether (which does not contribute to the present investigation very much under the assumptions to be made). We stress that the resulting dynamics on the state variable level are no longer of dimension 18 as in Chiarella, Flaschel and Zhu (2003), but now of dimension 19, since the law of motion of workers' debt (formerly workers' bond accumulation) now feeds back into the market for goods due to the dependence of workers' goods demand on the interest payments for their loans. The present model thus not only reinterprets the worker households' bond accumulation as debt accumulation and adjusts their behavioral equations to this new interpretation, but it also adds a feedback chain to the dynamics that was formerly missing (on which the possibility of the occurrence of debt (wage) deflation is based here).\footnote{See Chiarella, Flaschel and Semmler (2001a,b) for similar, but possibly much more severe situations of debt (price) deflation, there with respect to credit relationships between asset holders and firms.}

We start with a compact presentation (including brief comments on their contents) of the 19 laws of motion of the full model of this paper, and will present thereafter a 9D core case (with a unique interior steady state solution) of these dynamics to be used in the analysis that follows. These laws of motion around the steady state of the dynamics, appropriately grouped together and all in per unit of capital terms, are given by the following set of differential equations. As first group we consider here the quantity adjustment mechanisms with respect to the market for goods, concerning sales expectations $y^e$ and actual inventories $\nu$, and for the market for labor, concerning the employment policy of firms, $l^{we}$, and also concerning the evolution of full employment labor intensity $l^e$ (both in efficiency units) and of the stock of housing (everything per unit of the capital stock of firms):\footnote{We denote by $\gamma$ the given trend growth rate in the world economy which is used as trend term in the investment equations that apply to the domestic economy. Note also that we follow Chiarella and Flaschel (1999a) in assuming a fixed proportions technology in the following with a given potential output - capital ratio $y^p$ and a (temporarily) given output/ labor ratio $x$ that is subject to Harrod neutral technological change at rate $\dot{x} = n_t$.}

\[
\begin{align*}
y^e & = \beta_y(y^d - y^e) + (\gamma - (g_k - \delta))y^e, \\
\dot{\nu} & = y - y^d - (g_k - \delta)\nu, \\
l^{we} & = \beta_l(l^{de} - l^{we}) + [\gamma - (g_k - \delta)]l^{we}, \\
\dot{l^e} & = \gamma - (g_k - \delta), \\
\dot{k}_h & = g_h - \delta_h - (g_k - \delta).
\end{align*}
\]

The first of these five laws for quantity movements describes the adjustment of sales expectations $y^e$ in view of the observed expectational error $y^d - y^e$ based on currently realized sales $y^d$, augmented by a term that takes account of the fact that this adjustment is occurring in a growing economy and is expressed in intensive form. Next, inventories $\nu$ (per unit of capital) change according to the gap between actual output $y$ and actual sales $y^d$, again reformulated such that growth of the capital stock, the measurement base for the considered intensive form variables, is taken into account. Employment of firms, $l^{we}$, is changed in order to reduce the discrepancy that exists in each moment of time between the actual employment $l^{de}$ of the employed and their normal employment, here measured by $l^{we}$ (everything in efficiency units due to the assumed form of technical change). The growth rate of the factor endowment ratio $l^e = L^e/K$ (in efficiency units) is simply given by the difference between the natural rate of growth (including Harrod neutral technical change) and the growth rate of the capital stock $g_k - \delta$. Similarly, the growth rate of the housing stock (per unit of the capital stock of firms) is given by the difference of the corresponding net accumulation rates. We will assume in
the analytical treatment of the model that employment of firms adjusts with infinite speed, \( \beta_t = \infty \) to the actual employment of their labor force so that there is no over- or under-employment of this labor force in this situation. There thus remain only four laws of motion of quantities of which the first two will in addition be replaced by a static relationship in the further evaluation of the model.

Next we consider the nominal dynamics in the real sector of the economy which are described by four dynamical laws. Note here that the laws of motion for wages, \( \omega^e \), in efficiency units, and prices, \( p_y \), are now formulated independently from each other\(^{13}\) and show that reduced form Phillips curves\(^{14}\) (exhibiting only price inflation) are generally not as simple as is often assumed in the literature:\(^{15}\)

\[
\begin{align*}
\dot{\omega}^e &= \pi^t + \kappa[\beta_{\omega}(l^{de}/l^e - \bar{V}) + \kappa_\omega \beta_p(y/y^p - \bar{U})], \\
\dot{p}_y &= \pi^t + \kappa[\kappa_p(\beta_{\omega}(l^{we}/l^e - \bar{V}) + \beta_p(y/y^p - \bar{U})], \\
\pi^t &= \beta_{\pi t}(\alpha_{\pi t}(\pi^t - \pi^t) + (1 - \alpha_{\pi t})(0 - \pi^t)), \\
\dot{p}_h &= \beta_h\left(\frac{\kappa_\omega}{k_h} - \bar{U}_h\right) + \kappa_h \dot{p}_y + (1 - \kappa_h)\pi^t.
\end{align*}
\]

As already noted we now use reduced form Phillips curves for wage inflation \( \dot{\omega}^e \) and price inflation \( \dot{p}_y \), which both depend on the demand pressures in the markets for labor as well as for goods, \( y/y^p - \bar{U} \). The change of the rate of inflation expected over the medium run, \( \pi^t \), is determined as a weighted average of adaptively formed expectations and regressive ones (which realize that the steady state rate of inflation is zero in the present model). Finally, the inflation rate for housing services depends on the demand pressure term \( \frac{\kappa_\omega}{k_h} - \bar{U}_h \) in the market for these services,\(^{16}\) and on actual and perceived cost push expressions, here simply based on a weighted average concerning the inflation rate of domestic output. We shall assume in the following that \( \beta_{\pi t} = 0 \) holds and thus will not consider the role of inflationary expectations in this paper (which would add extra instability to the model if the price level is adjusting with high speed and if \( \beta_{\pi t} \) is chosen sufficiently large, the so-called Mundell-effect of Keynesian type models which include a wage price sector).

Next follow the dynamical laws for long-term bond price dynamics and exchange rate dynamics (including expectations) which basically formulate a somewhat delayed adjustment towards interest rate parity conditions and are supplemented by heterogeneous expectations formation (of partially adaptive and partially perfect type). Note that perfect foresight, concerning the proportion \( 1 - \alpha_t \) of market participants, can be removed from explicit representation as it coincides with the left hand side of the corresponding price adjustment equation,

\(^{13}\)We do not consider in the present simplified form of the model payroll taxes and value added taxes which helps to simplify the notation.

\(^{14}\)Here, \( \kappa \) denotes the expression \( 1/(1 - \kappa_\omega \kappa_p) \in (0, \infty) \).

\(^{15}\)Such disentangled laws of motion for nominal prices and wages are obtained from their originally independent presentation, see the preceding section, by solving the two linear equations of this section with respect to the variables \( \dot{\omega}^e - \pi_t, \dot{p}_y - \pi_t \) which implies the expressions shown below, which make use of both of our measures of demand pressure on the market for labor \textit{and} for the goods market (and on expected medium-run inflation). It is intuitively obvious that the removal of wage or price inflation cost-push pressure, \( \dot{\omega}^e, \dot{p}_y \) from the original form of the price or wage dynamics must imply that both the goods and the labor market expressions will be present in the resulting disentangled Phillips curves which thus are in a significant way more general than the ones usually considered in the theoretical or applied literature on price Phillips curves (unless one assumes – as some sort of Okun’s law – that all demand pressure variables used are positive multiples of each other).

\(^{16}\)Where \( \frac{\kappa_\omega}{k_h} \) represents the rate of capacity utilization on this market and \( \bar{U}_h \) the corresponding NAIRU level.
giving rise to the fractions in front of these adjustment equations, see Chiarella and Flaschel (1999a,b, 2000a) for details):

\[
\hat{p}_b = \frac{\beta_{pb}}{1 - \beta_{pb}(1 - \alpha_s)}[(1 - \tau_c)r_l + \alpha_s \pi_{bs} - (1 - \tau_c)\pi], \quad r_l = 1/p_b,
\]

\[
\hat{\pi}_{bs} = \beta_{\pi_{bs}}(\hat{p}_b - \pi_{bs}),
\]

\[
\hat{c} = \frac{\beta_c}{1 - \beta_c(1 - \alpha_s)}[(1 - \tau_c)r^*_l + \alpha_s \epsilon_s - ((1 - \tau_c)r_l + \pi)], \quad r_l = 1/p_b,
\]

\[
\hat{\epsilon}_s = \beta_{\epsilon_s}(\hat{c} - \epsilon_s).
\]

These laws of motion are not made use of in the following since we shall assume in this paper that the required rate of return \(\rho^*\) used in the description of investment of firms and asset holders is a given magnitude, measured by the world rate of interest \(r^*_l\) and since we will also assume that the measure for the tightness of monetary policy, \(\tau_l - \tau\), is not involved in the formation of these investment plans (by setting the corresponding coefficients of the investment functions of the preceding section equal to zero). A further assumption needed to avoid any further discussion of these laws of motion will be provided when the next block of laws of motion, concerning the government sector, is considered. Note with respect to the above equations that the literature generally only considers the border case where \(\alpha_s = 0\) is used in conjunction with infinite adjustment speeds on the two considered markets. This gives rise to two interest parity conditions which, when coupled with myopic perfect foresight on bond price and exchange rate movements, gives rise to a situation of knife edge instability, which is then stabilized by means of the jump variable technique.

The next set of dynamical laws concerns the evolution of short-term and long-term debt of the government (the issuing of which is here governed by fixed proportions \(\alpha^g_b, 1 - \alpha^g_b\)), its wage and import taxation policy and the interest rate policy of the central bank. Note here that we continue to use the letter \(b\) to denote government debt per unit of capital and that its short term debt \(b_g\) must now be indexed by \(g\) since there is also the debt of worker households (which we here denote by \(d_w\) in order to stress their importance for the present investigation):\(^{17}\)

\[
\hat{b}_g = \alpha^g_b[gy^e + rb_g + b^l - t^a - t^c + w^a] - (\hat{p}_g + g_k - \delta)b_g,
\]

\[
\hat{b}^l = (1 - \alpha^g_b)[gy^e + rb_g + b^l - t^a - t^c + w^a]/p_b - (\hat{p}_g + g_k - \delta)b^l,
\]

\[
\hat{\tau}_w = \alpha_{\tau_l}(\frac{d_g}{d_g - 1}), \quad d_g = \frac{b_g + p_b b^l}{y^c},
\]

\[
\hat{\tau}_m = \alpha_{\tau_m}(\frac{p_{m}^x}{p_{m}^x} - (1 + \tau_m)p_{m}^j d), \quad x = x_y y, j^d = j_y y,
\]

\[
\hat{r} = -\beta_{r_1}(r - r^*_l) + \beta_{r_2}(\hat{p}_g - \tilde{\pi}) + \beta_{r_3}(y/y^p - \bar{U}), \quad \tilde{\pi} = 0.
\]

Since these laws of motion, up to the interest rate policy rule, are also suppressed by appropriate assumptions in the analysis that follows we here only briefly state that the first two are immediate consequences of the government budget constraint (based in particular on various sources of tax income, now diminished by subsidies that concern the interest payments of worker households), that wage taxation is here adjusted in the direction of a target ratio of government debt, \(d_g\), and that import taxes are adjusted in a way that ensures a balanced trade account in the steady state (which greatly simplifies the calculation of the steady state

\(^{17}\)The expressions \(t^a, t^c, w^a\) represent tax payments out of wages and profits and transfer payments of the government that will be of no importance in the core 9D dynamics that is the focus of this paper.
of the model). The interest rate policy rule \( \hat{r} \) is of interest however since it could be of help in counteracting accelerating debt (wage) deflation, by lowering nominal interest rates in situations of depressed activity levels and price deflation. This rule has already been explained in the preceding section. We assume in the following that the wage tax rate is not adjusted at all \( (\alpha_{w} = 0) \) and set equal to the steady state value of the general 19D model and that the import taxes are adjusted with infinite speed \( (\alpha_{m} = \infty) \). These two assumptions imply in the reduced formulation of the model given below that the evolution of government debt does not feed back into the core dynamics of the model and that the exchange rate does not matter for them (also due to the assumption of given world market prices for both imports and export commodities).

As 19th law of motion, which is not new to the model but is now interacting with its core dynamics due to its feedback on the spending behavior of workers households, we finally have:

\[
d_w = (c_1 + c_2 - 1)y_w^D - (\hat{p}_y + g_k - \delta)d_w,
\]

which determines the evolution of the debt to capital ratio of workers' indebtedness to the other type of households, the asset holders. This law of motion, together with the possibility of housing cycles due to the investment in housing and the rate of return that characterizes the housing sector and the possibility of price deflation, will be the focus of interest of the present paper. Note here that the debt to capital ratio \( d_w \) influences its rate of change negatively as far as the term based on the disposable income of workers is concerned, since this income depends negatively on this ratio and since the sum of workers' marginal propensities to spend has been assumed to be larger than one. However, due to this situation, we also have that aggregate demand, economic activity and thus goods price inflation depends negatively on \( d_w \), which introduces a positive dependence between the rate of change of this ratio and its level. This is indeed the partial debt deflation mechanism of the model we have already described in the preceding section. Note here that we do not yet have credit rationing in the model which would establish a further channel by which aggregate demand may be reduced in deflationary episodes.

Summarizing we can thus state that we will basically consider the following subdynamics of the general 19D dynamics in the present paper (in the next section) and will do this by making use of further simplifications of these dynamics that allow for the possibility of an analytical treatment:

\[
\begin{align*}
y^e &= \beta_y^e(y^d - y^e) + (\gamma - (g_k - \delta))y^e, \quad (17) \\
\dot{v} &= y - y^d - (g_k - \delta)v, \quad (18) \\
\dot{\ell}^e &= \gamma - (g_k - \delta), \quad (19) \\
\dot{h}_h &= g_h - \delta_h - (g_k - \delta), \quad (20) \\
\dot{w}^e &= \kappa[(1 - \kappa_p)\beta_w(l^{de}/I^e - \bar{V}) - (1 - \kappa_w)\beta_p(y/y^p - \bar{U})], \quad (21) \\
\dot{p}_y &= \kappa[\kappa_p\beta_w(l^{de}/I^e - \bar{V}) + \beta_p(y/y^p - \bar{U})], \quad (22) \\
\dot{p}_h &= \beta_h(\frac{c_h^p}{k_h} - \bar{U}_h) + \kappa_h\dot{p}_y, \quad (23) \\
\dot{d}_w &= (c_1 + c_2 - 1)y_w^D - (\hat{p}_y + g_k - \delta)d_w, \quad (24) \\
\dot{r} &= -\beta_{r_1}(r - r_1^*) + \beta_{r_2}\hat{p}_y + \beta_{r_3}(y/y^p - \bar{U}). \quad (25)
\end{align*}
\]
Note we use in these laws of motion now the real wage \( \omega^e = w^e/p_y \) in the place of the nominal wage. These laws of motion make use of the following supplementary intensive form definitions and abbreviations (which are not explained here in detail since we only provide the essential features of the modelling approach of Chiarella and Flaschel (1999a,b,2000a) to be investigated in the following): \(^{18}\)

\[
\begin{align*}
y &= y^e + \beta_n(\beta_n y^e - \nu) + \gamma \beta_n y^e, & \text{output per unit of capital}, \\
l^{de} &= l^n, & \text{the labor coefficient in efficiency units}, \\
y^D_w &= (1 - \tau_c)\omega^e l^{de} - (1 - \tau_c) r d_w, & \text{real disposable income of workers}, \\
c^w_c &= c_1 y^D_w, & \text{goods consumption of workers (including the purchase of houses)}, \\
c^w_h &= p_y c_2 y^D_w / p_h, & \text{housing services consumption of workers}, \\
\rho^e &= y^e - \delta - \omega^e l^{de}, & \text{the expected real rate of profits of firms}, \\
g_k &= \alpha^k_1(1 - \tau_c)(\rho^e - \tau^*_t) + \alpha^k_3(\gamma y^p - \bar{U}) + \gamma + \delta, & \text{gross investment of firms}, \\
\rho_h &= (p_h/p_y) c^w_h / k_h - \delta_h, & \text{real rate of profit for housing investment}, \\
g_h &= \alpha^h_1(1 - \tau_c)(\rho_h - \tau^*_t) + \alpha^h_3(c^w_h / k_h - \bar{U}_h) + \gamma + \delta_h, & \text{gross investment in housing}, \\
y^d &= c^w_c + g_k + g_h k_h + g y^e, & \text{aggregate demand (including government demand } g y^e), \\
\tau_w &= 1 - (y^D_w + (1 - \tau_c) r_d w) / (\omega^e g d^e).
\end{align*}
\]

Insertion of these equations into the above 9 laws of motion gives an explicit system of nine autonomous nonlinear differential equations in the 9 state variables of the model shown in eq.s (17) - (25) we arrived at in this section. Note that we have removed pension payments and unemployment benefits from the above presentation of the model and that the evolution of price levels is subject to zero-root hysteresis, since it depends on historical conditions due to our assumptions on the interest rate policy rule of the central bank and the accompanying assumption of costless cash balances (during each trading period) for the four agents of the model, see Chiarella and Flaschel (1999a) for details.

We present next the 9 steady state values of the model (including further definitional equations that are needed for their full determination). All of these values should have an index ‘o’ (denoting their steady state character) when used for programming purposes. To not overload the notation however we do not add this index to the following list of steady state values. Note that the steady state values of level magnitudes are all expressed in per unit of capital form and if necessary in efficiency units, see Chiarella, Flaschel and Zhu (2003) for the details in the case of the 18D core model. Note also that we have now debt of workers and of the government in the steady state of the model and that we therefore denote their actual and steady debt - capital ratios by choosing appropriate indexes in both cases: \( d_w, d_g. \) \(^{19}\) Note finally that the steady state is parametrically dependent on a given output price level \( p_y \) which is not determined by the model (due to the Taylor type interest rate policy pursued by the central bank) and thus can be supplied from the outside in an arbitrary fashion: \(^{20}\)

\[
y^e = \frac{y}{1 + \gamma \beta_n d^e}, \quad y = y^p \bar{U}, \quad (26)
\]

\(^{18}\)Note that output \( y \) differs from expected sales \( y^e \) due to voluntary inventory investments of firms.

\(^{19}\)\( d_g \) the aggregate debt of the government sector.

\(^{20}\)The inflation target of the central bank \( \pi \) is a zero rate of inflation here (which is not true for actual central bank behavior in general). This implies in the present model that steady state inflation will be zero, too, which in turn implies that the levels of nominal magnitudes are fixed magnitudes in the steady state (in efficiency units solely as far as nominal wages are concerned).
\[ \nu = \beta_n y^e, \quad (27) \]
\[ l^e = \frac{l^{de}}{\bar{V}}, \quad l^{de} = r^*_p y, \quad (28) \]
\[ k_h = \frac{(1 - g)y^e - (\gamma + \delta)}{c_1(r^*_t + \delta_h)/c_2 + \gamma + \delta_h}, \quad c_h^w = \bar{U} k_h, \quad (29) \]
\[ \frac{p_h}{p_y} = \frac{(r^*_t + \delta_h)/\bar{U}_h}{(r^*_t + \delta_h)} \quad (30) \]
\[ y_w^D = \frac{(p_h/p_y)k_h\bar{U}_h/c_2}{y_w^D} + (1 - \tau_r)r^*_td_w, \quad (32) \]
\[ \tau_w = 1 - y_w^D/(\omega^{sde}), \quad (33) \]
\[ \omega^e = \frac{y^e - \delta - r^*_t}{l^{de}}, \quad w^e = \omega^e p_y, \quad (34) \]
\[ p_y = \text{determined by initial conditions}, \quad (35) \]
\[ p_h = p_y (r^*_t + \delta_h)/\bar{U}_h, \quad (36) \]
\[ r = r^*_t, \quad (37) \]
\[ d_w = \frac{c_1 + c_2 - 1}{\gamma} y_w^D. \quad (38) \]

Note that \( d_w \) is positive in the steady state due to our assumption \( c_1 + c_2 > 1 \) so that workers' debt grows in line with the capital stock in the steady state (as do workers' interest payments). Note also again that the steady state is inflation-free due to our assumption on monetary policy and that nominal wages rise with labor productivity in the steady state.

Equation (26) gives the steady state solution of expected sales \( y^e \) per unit of capital \( K \) (and also of output \( y \) per \( K \)) as determined by the desired utilization rate of firms and the inventory policy they have to adopt due to demand growth in the steady state. Eq. (27) provides on this basis the steady inventory-capital ratio \( N/K \) which says that inventories (to be produced in addition to actual sales) must grow at the same speed as the capital stock. Eq. (28) represents (in efficiency units) the amount of workforce (per \( K \)) employed by firms in the steady state as well as full employment labor intensity which is larger than actual labor intensity (in efficiency units) due to the assumed NAIRU rate of employment \( \bar{V} < 1 \). The last expression for the quantity side of the model, in eq. (29), provides the steady value of the housing capital stock per unit of the capital stock of firms which is obtained on the basis of the calculation of the income magnitudes there shown and the debt to capital ratio of worker households to be determined below. Eq. (34) concerns the wage level (in efficiency units), real and nominal, to be derived from the steady state value for the rate of profit which is given by the world rate of interest \( r^*_t \) here. Note that all nominal magnitudes (up to nominal wages) exhibit no long run trend and that the steady price level of output prices \( p_y \) is not determined by the model. As remaining nominal magnitude we have the price level \( p_h \) for housing rents (in eq. (36)), to be calculated from the uniform rate of interest \( r^*_t \) of the economy in the steady state (which also characterizes the rate of return in the housing sector). There follows the steady value of the debt to capital ratio \( d_w \) of workers, the only debt ratio to be considered in the following due to the assumption of a given wage tax rate. With respect to the public sector, there is finally the interest rate policy rule of the central bank, which due to its formulation implies that the interest rate on short-term government debt must also settle down at the given foreign rate, \( r^*_t \), in the steady state. Again, the new equation is eq. (37), where the steady state debt - capital ratio of workers is easily obtained from their budget constraint of workers and is positive if and only if \( c_1 + c_2 > 1 \) holds true. This closes the presentation of the interior steady state solution of our reduced 9D dynamical model. We note that the capital to debt ratio of workers rises with \( c_1 + c_2, y_w^D, \tau_c \) and falls with \( \gamma, r^*_t \).
4 2D, 3D and 5D subcases of an integrated 6D real subdynamics

Using an approach similar to that in Chiarella, Flaschel and Zhu (2003) we simplify the 9D dynamics of the preceding section further, by assuming in the place of the delay chain that is based on a goods market disequilibrium adjustment process of the type

\[ y^d = c_1 y^D + \alpha^k (\rho^e - r^*_1) + \alpha^k (y^p - \bar{U}) + \gamma + \delta \rightarrow y^e \rightarrow y \rightarrow y^d, \]

a simplified static and linearized\(^{21}\) equilibrium relationship of the kind:

\[ y^d = y^e = y = y(\omega^e, d_w) = \bar{U} y^p + d_1 (\omega^e - \omega^e_0) + d_2 (d_w - d^o_w), \quad d_1, d_2 \leq 0. \]

This relationship between output, real wages and real debt will be used in the following as a shortcut for the delay chain of the general case (and its richer concept of aggregate demand and its determinants) in order to study the effects of wage and price inflation and deflation on debt and real wages in a significantly simplified version of the 9D model. Note that this shortcut of the originally delayed quantity adjustment process of Metzlerian type requires that the steady state value of this function \( y \) must be equal to \( y^p \bar{U} \) in order to get a steady state solution for this 7D simplification of the 9D dynamics. Note also that we concentrate in this presentation of goods market equilibrium on the effects of real wages and debt ratio increases which both are assumed to have a (non-)negative influence on goods market behavior, i.e., in the case of real wages that the resulting decrease in investment demand outweighs the implied increase in consumption.

Let us furthermore assume that \( \kappa_h = 1 \) holds, i.e., the cost-push term in the dynamics of rental prices is given solely by the current rate of inflation on the market for goods. This assumption allows us to reduce the dynamics to a consideration of relative prices only, namely the real wage (as before) and the real rental price. On the basis of this assumption and the above short-cut for goods market dynamics the dynamical system to be investigated in the following reads:

\[
\begin{align*}
\dot{\gamma} &= \gamma - (g_k - \delta), \\
\dot{\bar{U}} &= g_k - \bar{U} - (g_k - \delta), \\
\dot{\omega^e} &= \kappa[(1 - \kappa_p)\beta_w(t^{de}/t^e - \bar{V}) - (1 - \kappa_w)\beta_p(y/y^p - \bar{U})], \\
\dot{\pi_h} &= \beta_h(\pi^w_{h^*} - \bar{U}), \quad \pi_h = p_h/p_y, \\
\dot{d_w} &= (c_1 + c_2 - 1) y^D_w - (\kappa p\beta_w(t^{de}/t^e - \bar{V}) + \beta_p(y/y^p - \bar{U})] + g_k - \delta d_w, \\
\dot{r} &= -\beta_{r_1}(r - r^*_1) + \beta_{r_2}\kappa[\kappa_p\beta_w(t^{de}/t^e - \bar{V}) + \beta_p(y/y^p - \bar{U})] + \beta_{r_3}(y/y^p - \bar{U}).
\end{align*}
\]

now with the static relationships:

\[
\begin{align*}
y &= \bar{U} y^p + d_1 (\omega^e - \omega^e_0) + d_2 (d_w - d^o_w), \quad d_1, d_2 \leq 0, \\
l^{de}_f &= l^e_y y.
\end{align*}
\]

\(^{21}\)Around the interior steady state of the model, given by \( \omega^e_0, d^o_w. \)
Neglecting the interest rate policy (AQP of the monetary authority for the moment (by setting the corresponding adjustment parameters equal to $\infty, 0$ and $0$, which implies $r = r^*$) we have that interest payments of workers are based on a given rate of interest. The resulting 5D dynamics are then based on the growth laws for full employment labor intensity, housing capital per unit of capital, real wages and real rental prices and finally as the financial variable the debt to capital ratio of worker households.

The underlying interior steady state solution of these 5D dynamics (and also of the 6D dynamics) is characterized by:

\[
\begin{align*}
y_w^D &= (1 - \tau_w)\omega^e l^{de} - (1 - \tau_c)rd_w, \\
c_h^w &= c_2(1 - \tau_w)y_w^D/\pi_h, \\
\rho &= y - \delta - \omega^e l^{de}, \\
g_k &= \alpha^k(1 - \tau_c)(\rho - r^*_l) + \alpha^k(y/y^p - \bar{U}) + \gamma + \delta, \\
\rho_h &= \pi_h c_h^w/k_h - \delta_h = c_2(1 - \tau_w)y_w^D/k_h - \delta_h, \\
g_h &= \alpha^h(1 - \tau_c)(\rho_h - r^*_l) + \alpha^h(ch^w/k_h - \bar{U}_h) + \gamma + \delta_h, \\
\tau_w &= 1 - y_{w1}^D/(\omega^e l^{de}), \quad y_{w1}^D = y_w^D + (1 - \tau_c)r^*_ld_w.
\end{align*}
\]

The above 5D dynamical system can be subdivided further into a natural real growth cycle model of the Goodwin (1967), Rose (1967) type concerning the interaction of capital accumulation and income distribution and into a 3D dynamical system where we can study the interaction of the growth of the housing stock (for rental purposes) with real rental price adjustments and the debt to capital ratio of workers used in particular to finance their investment into their own stock of houses.

In order to obtain the first set of subdynamics we have to assume in addition to the assumptions already made that $d_2 = 0$ holds true, i.e., there is no debt effect with respect to the state of the goods market used in the following dynamical subsystem for describing real wage and investment dynamics. The resulting dynamics in the full employment labor intensity $l^e$ and the real wage $\omega^e$ is basically of Goodwin (1967) growth cycle type augmented by Rose (1967) type effects of the real wage on its rate of change (as we shall see in detail below):
\[ \dot{e} = -\left(\alpha_1^e(1 - \tau_c)(\rho - r^*_1) + \alpha_2^e(y/y^p - \bar{U})\right), \quad (45) \]
\[ \dot{\omega}^e = \kappa(1 - \kappa_p)\beta_\omega(t^d - \bar{V}) - (1 - \kappa_w)\beta_p(y/y^p - \bar{U}), \quad (46) \]

We now have as remaining static relationships:
\[
\begin{align*}
y &= \bar{U}y^p + d_1(\omega^e - \omega^e_0), \quad d_1 < 0, \\
l^d &= l_y^e, \\
\rho &= y - \delta - \omega^e l^d.
\end{align*}
\]

The assumption \( d_1 < 0 \) represents what we will call a negative Rose-effect in the following since it will imply that wage flexibility is stabilizing and price flexibility destabilizing just as in the original contribution of Rose (1967).

The steady state of these 2D dynamics is characterized by
\[
\begin{align*}
y &= y^p \bar{U}, \\
l^d &= l_y^e, \\
l^e &= l^d / \bar{V}, \\
\rho &= r^*_1, \\
\omega^e &= \frac{y - \delta - r^*_1}{l^d}.
\end{align*}
\]

Assuming on the other hand no fluctuations in the capital stock of firms, \( \dot{K} = g_k - \delta = \gamma \) and in the real wage of workers, \( \omega^e = \omega^e_0 \) by contrast gives rise to interacting dynamics between the stock of houses, \( k_h = K - h/K \), offered for rent on the market for housing services, the real rental price of housing services, \( \pi_h = p_h/p_y \), and the real debt to capital ratio \( d_w = D_w/(\rho_y K) \):\(^{22}\)

\[
\begin{align*}
\dot{k}_h &= \alpha_1^h(1 - \tau_c)(\rho_h - r^*_1) + \alpha_2^h(c_h^w/k_h - \bar{U}_h), \quad (47) \\
\dot{\pi}_h &= \beta_h(c_h^w / k_h - \bar{U}_h), \quad (48) \\
\dot{d}_w &= (c_1 + c_2 - 1)y_w^D - (\kappa_1\kappa_p\beta_\omega(t^d / l_y^e - \bar{V}) + \beta_p(y/y^p - \bar{U})) + \gamma)d_w, \quad (49)
\end{align*}
\]

here with the static relationships:
\[
\begin{align*}
y &= \bar{U}y^p + d_2(d_w - d^2_w), \quad d_2 < 0, \\
l^d &= l_y^e, \\
y_w^D &= (1 - \tau_w)\omega^e_0 l^d - (1 - \tau_c)r^*_1 d_w, \\
c_h^w &= c_2 y_w^D / \pi_h, \\
\rho_h &= \pi_h c_h^w / k_h - \delta_h, \quad (50)
\end{align*}
\]

which reduces further to a 2D system where the evolution of debt does not matter if \( c_h^w = \text{const} \) is assumed.

\(^{22}\)Note here that \( \dot{K} = \gamma \) implies \( l^e = l_y^e \) if we start from the steady state and if shocks only occur in the state variables of the presently considered dynamics.
The interior steady state of these 3D dynamics is characterized by

\[
\begin{align*}
y &= y^p \bar{U}, \\
\bar{t}_{de} &= \bar{t}_{de}^p, \\
\rho_h &= \tau_1^*, \\
\pi_h &= (\tau_1^* + \delta_h)/\bar{U}_h, \\
d_w &= \frac{c_1 + c_2 - 1}{\gamma + (c_1 + c_4 - 1)(1 - \tau_c)} y_{w1}^D, \\
c_h^w &= c_2 y_{w1}^D/\pi_h, \\
k_h &= c_h^w/\bar{U}_h.
\end{align*}
\]

Note that the steady state values can be obtained from the above dynamics in this order and that these calculations in particular imply that the excess demand situations underlying the \(\beta_w, \beta_p\)-terms in the dynamics (47) – (49) are both zero in the steady state which in particular again implies that the price level is stationary in the steady state. Note also that this implies that the steady state value of \(d_w\) is uniquely determined as is claimed above.

We stress that the study of these partial subdynamics are not to be considered as being completely specified from the economic point of view, but should be viewed as an approach, eventually leading back to the fully specified 9D dynamics, that generates propositions of the as if variety, in the case of the above 3D dynamics, as if the debt effect on output can be considered as the one that dominates the outcome of the interaction of aggregate demand, sales expectations and output decisions of firms on the market for goods (as formulated for the full 9D dynamics).

With respect to this latter dynamics we now have as the first of our propositions:

**Proposition 1**

Assume that \(c_h^w\) is fixed at its steady state value (\(\bar{C}_h^w = \gamma\)). Then: The steady state of the dynamics (47), (48) is globally asymptotically stable for all positive starting values for \(k_h, \tau_h\), i.e., all trajectories in the positive orthant of \(\mathbb{R}^2\) converge to the steady state values of \(k_h, \tau_h\) shown above.

**Proof:** Concerning the Jacobian or linear part of the growth dynamics (47), (48), i.e., for the matrix of partial derivatives of the right hand side of (47), (48), paying no attention to the fact that we have growth rates in the place of time derivatives on the left hand side, we get at all positive tuples \((k_h, \tau_h)\) the qualitative expression:

\[
J = \begin{pmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{pmatrix} = \begin{pmatrix}
- & + \\
- & 0
\end{pmatrix}
\]

We thus in particular have trace \(J < 0\), det \(J > 0\) and \(J_{12}J_{21} \neq 0\). As shown in Flaschel (1984) these three conditions imply the assertion due to a particular application of Olech’s theorem on global asymptotic stability.

\(\Box\)

We observe with respect to proposition 1 and its proof that these dynamics would be of the Goodwin (1967) center type were there not the negative (stabilizing) influence of the state variable \(k_h\) on its own evolution.

The next proposition adds the influence of the debt to capital ratio to the dynamics just considered (via the effect this ratio has on the output of firms) and it of course adds also the
budget law that determines the evolution of real debt per unit of capital. Contrary to what one
might expect, we here find that these additional aspects do not endanger the stability result
just obtained, if price adjustment is sufficiently sluggish, which due to the increased dimension
of the dynamics can now however only be shown for an appropriately chosen neighborhood of
the steady state. Yet, the included debt effects will be destabilizing if the adjustments caused
by goods and labor market disequilibrium in the wage - price module of the model become
sufficiently pronounced.

**Proposition 2**

The steady state of the dynamics (47) - (49) is locally asymptotically stable, if the
parameters \( \beta_p \) and \( \kappa_p \) (or \( \beta_w \)) are chosen sufficiently small (such that \( J_{33} < 0 \) holds,
see the proof). Conversely, this steady state will be unstable if the parameters \( \beta_p \)
or \( \beta_w \), the latter for \( \kappa_p > 0 \), are chosen sufficiently large (such that \( J_{33} > 0 \) holds).

**Proof:** Due to the continuity of eigenvalues with respect to parameter changes we only
need to consider the assertion of local asymptotic stability in the case where \( \beta_p = \kappa_p = 0 \) holds.
The Routh-Hurwitz theorem then in particular states that all eigenvalues of the considered
Jacobian will have negative real parts if the Routh-Hurwitz coefficients fulfill
\( a_1 = - \text{trace} J > 0 \), \( a_2 = J_1 + J_2 + J_3 > 0 \), \( a_3 = - \det J > 0 \) and finally
\( a_1 a_2 - a_3 > 0 \), a situation which, as just stated, is not changed if small variations of the parameters \( \beta_p \) and \( \kappa_p \) away from zero are allowed for. Note here that the coefficient \( a_2 \)
represents the sum of the principal minors of order 2 of the considered Jacobian \( J \).

It is easy to show that the trace \( J_{11} + J_{22} + J_{33} \) of the Jacobian \( J \) must be negative in
the assumed situation, since all auto-feedbacks of the system (47) - (49) are negative, i.e.,
all three coefficients making up the trace are negative here. Concerning \( J_1, J_2 \) and \( J_3 \), whose
indices refer to the row and column not considered in these subdeterminants, one also gets
immediately (from what has just been shown for the trace) that both \( J_1 \) and \( J_2 \) must be
positive, since \( J_{31} \) and \( J_{32} \) are both zero so that only multiplication with respect to elements
from the diagonal of \( J \) is involved here. With respect to \( J_3 \) one gets furthermore that
the dynamical law for \( k_h \) can be reduced to

\[
\dot{k}_h = a_1^h (1 - \tau_c) \pi_h,
\]

without changing the sign of \( J_3 \), by making use of the linear dependencies that exist with
respect to the second dynamical law (for \( \pi_h \)) that is involved in the calculation of \( J_3 \). From
this we see qualitatively that

\[
J_3 = \begin{pmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{pmatrix}
= \begin{pmatrix}
0 & + \\
- & -
\end{pmatrix} > 0,
\]

as was claimed above.

Calculating \( \det J \) one can use such a linear dependency in addition in order to arrive at

\[
\det J = \begin{vmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{vmatrix}
= \begin{vmatrix}
0 & + & 0 \\
- & - & - \\
0 & 0 & -
\end{vmatrix}.
\]

This not only shows that \( \det J \) must be negative, but also that \(- \det J \) must be equal to or
smaller than \( J_1 (-J_{33}) \) which finally gives that also \( a_1 a_2 - a_3 > 0 \) must hold true, since \( a_1 a_3 \)
is based on positive expressions throughout.
Concerning the second assertion, on instability, one simply has to note that the third law of motion implies for positive parameters $\beta_p$, $\kappa_p$ and $\beta_w$ for the entry $J_{33}$ of $J$ at the steady state:

\[
J_{33} = (c_1 + c_2 - 1)((1 - \tau_w)\omega^p \alpha_0^p d_2 - (1 - \tau_o)\gamma) + \kappa[\kappa_p\beta_w\alpha_0^p(-d_2)/\alpha_0^p + \beta_p(-d_2)/\alpha_0^p]d_w.
\]

This immediately shows that trace $J$ can be made as positive as is desired by choosing either $\beta_p$ or $\beta_w$, the latter for $\kappa_p > 0$, sufficiently large, since $-d_2 > 0$ and $d_w > 0$ hold.

Note finally that in the present formulation of the dynamics (47) – (49) we always have that the third law of motion is independent of the other ones, so that $J_{31}$ and $J_{32}$ are always zero (which simplifies the above stability arguments further). The benchmark for asymptotic stability therefore is the situation where $J_{33} < 0$ holds true and instability in the present situation is therefore solely due to the law of motion for the debt to capital ratio $d_w$. 

In view of this last observation on the (in)stability of the model we should however stress here that we have approached this proposition and its proof from a slightly more general perspective than was really necessary in order to indicate how it can be applied to more general situations than considered above. Assume for example that the marginal propensities to consume $c_1$ and $c_2$ both depend positively on the relative price for housing services $\tau_h$ such that real expenditure on housing services $c_h^w$ depends negatively on $\tau_h$ (but, as assumed, not nominal expenditures on these services). Assume also that domestic output $y$ depends positively on $\tau_h$. The proposition 2 basically also holds true in such an augmented situation, since trace $J$ stays negative, since linear dependencies again imply that $J_1$, $J_2$ and $J_3$ are all positive and since $\det J$ can in this way be reduced to the form

\[
\det J = \begin{vmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{vmatrix} = \begin{vmatrix}
0 & + & 0 \\
- & 0 & 0 \\
0 & 0 & -
\end{vmatrix},
\]

which again shows that this determinant is negative and dominated by the positive expressions in $a_1a_2$. The situation of proposition 2 therefore can be generalized to cases where the third law of motion is no longer independent of the other two differential equations.

**Proposition 3**

The steady state of the dynamics (47) – (49), if locally asymptotically stable, is never globally asymptotically stable, but will be explosive in the debt to capital ratio if this ratio is chosen sufficiently large.

**Proof:** We know in the assumed situation that $J_{33} < 0$ holds true at the steady state. Considering the right hand side of eq. (49) it is, however, obvious from the preceding proof that there must be a second root of this equation (where $d_w = 0$ holds) and where $J_{33} > 0$ is true. This follows from the fact that the right hand side of this equation is a polynomial of order 2 in the state variable $d_w$ with a positive coefficient in front of the $d_w^2$-term. To the right of this root, the debt to capital ratio will increase beyond any bound, since $d_w > 0$ is then given for all points in time. 

Let us now consider the other subdynamics (45) – (46) of the 5D system (39) – (43) where it is assumed that the rate of interest on the debt of workers is a given magnitude ($= \gamma^*$) and not subject to policy considerations by the central bank. Neglecting again the growth rate

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formulation of these dynamics, the Jacobian of the right hand side of this system reads for all points in the state space:

\[
J = \begin{pmatrix}
0 & -\alpha_1^k (1 - \tau_e) \rho_{\omega^e} - \alpha_3^k d_1/y^p \\
-\kappa (1 - \kappa_p) \beta_{\omega l} l^d e(l^e)^{-2} & \kappa (1 - \kappa_p) \beta_{\omega l} l^d_\omega d_1/l^e + \kappa (1 - \kappa_w) \beta_p (-d_1)/y^p
\end{pmatrix},
\]

with \( \rho_{\omega^e} \) given by \( d_1 (1 - \omega^e l^e_\omega) - l^e_\omega y < 0 \).

**Proposition 4**

Assume that \( \beta_p = 0 \) (or \( \kappa_w = 1 \)) holds. The interior steady state of the dynamics (45) and (46) is globally asymptotically stable for all positive starting values \( l^e \) and \( \omega^e \), i.e., all trajectories in the positive orthant of \( \mathbb{R}^2 \) converge to this steady state in the current situation.

**Proof:** Concerning the Jacobian \( J \) just calculated we get in this case for all points of \( \mathbb{R}^2 \) the qualitative expression

\[
J = \begin{pmatrix}
0 & + \\
- & -
\end{pmatrix}.
\]

We thus in particular have \( J < 0 \), \( \det J > 0 \) and \( J_{12} J_{21} \neq 0 \) and thus obtain the asserted global asymptotic stability as in proposition 1 by an appropriate application of Olech's theorem on global asymptotic stability. \( \square \)

This method of proof cannot be applied in the case \( \beta_p > 0 \) since we then have opposing signs in the element \( J_{22} \) of the trace of \( J \) with respect to the \( \beta_w \) and \( \beta_p \) expressions. Trace \( J \) may therefore change its sign (for large \( l^e \) for example) in the considered state space, though it may be negative at the steady state and thus imply local asymptotic stability, but not global asymptotic stability. In view of this we define a critical value for the parameter \( \beta_p \) (in the case \( \kappa_w < 1 \), for the steady state) by the expression

\[
\beta_p^H = \frac{1 - \kappa_p \beta_{\omega l} l^e_\omega / l^e}{1 - \kappa_w y^p}.
\]

With respect to this value we then get:

**Proposition 5**

1. The interior steady state of the dynamics (45), (46) is locally asymptotically stable for \( \beta_p < \beta_p^H \).
2. It is unstable for \( \beta_p > \beta_p^H \).
3. At \( \beta_p^H \) there occurs a Hopf-bifurcation, i.e., the steady state loses its stability at this parameter value (in general) by way of the death of an unstable limit cycle or the birth of a stable limit cycle as this parameter is crossed from below.

**Proof:** We have \( J_{33} = 0 \) at the bifurcation point and \( < 0 \) (\( > 0 \)) to the left (to the right) of it, which proves the first two assertions since \( \det J > 0 \). The third assertion is a standard assertion in the case where \( \det J > 0 \) holds throughout at the steady state, see Chiarella and Flaschel (2000b, Ch.3) for example. \( \square \)

We observe here that assertion 3 also holds with respect to proposition 2 in a similar and more trivial way (though the there considered dynamics is formally seen to be of dimension three). In sum we therefore have that increasing price flexibility may be dangerous for asymptotic stability for two reasons, applying in two different subdynamics of the 5D dynamics.
of this section, due to its adverse effects on the debt to capital ratio (a Fisher debt effect) and due to its adverse effect on real wage adjustment (a Rose effect). We expect of course that these two destabilizing mechanisms are jointly present in the integrated 5D dynamics and thus do not overthrow economic intuition when brought together in a higher dimensional environment.

This is easily shown for the system (39) – (43) since the Jacobian $J$ of this growth rate system at the steady state reads with respect to the elements that depend on the parameter $\beta_p$:

$$
J = \begin{pmatrix}
\kappa(1 - \kappa_w)\beta_p(-d_1)/y^p & \kappa(1 - \kappa_w)\beta_p(-d_2)/y^p \\
\kappa\beta_p(-d_1)d_w/y^p & \kappa\beta_p(-d_2)d_w/y^p
\end{pmatrix}
$$

(50)

Note that this expression only applies to the steady state of the dynamics and that we have used in this respect in particular that inflation is zero in the steady state. Obviously the trace expressions and the instability arguments based on them in the case of the disentangled 2D and 3D dynamics considered beforehand apply again, showing that the trace of $J$ can be made positive if the parameter $\beta_p$ is chosen sufficiently large (the point where trace $J$ becomes zero, and positive thereafter, is however now not given by a simple expression).

**Proposition 6**

1. The interior steady state of the 5D dynamics (39) – (43) is locally asymptotically stable if it is assumed that $y^p_w$ depends positively on the real wage $\omega_e$, if the parameters $\beta_p, \beta_w$ and $\beta_h$ are sufficiently small, and if $\kappa_p$ is sufficiently close to 1.

2. Asymptotic stability gets lost by way of a Hopf-bifurcation, at least in the case where $\kappa_p < 1$ holds (no stabilizing real wage based Rose effect), if the parameter $\beta_w$ is sufficiently large.23

3. Increasing the parameter $\beta_p$ leads from a negative to a positive determinant of the Jacobian of the considered dynamics at the steady state, i.e., loss of stability need not occur via a Hopf-bifurcation as the parameter $\beta_p$ is increased, since real parts of eigenvalues may now become positive by a movement along the real line.

We observe that the assumption that $y^p_w$ depends positively on the real wage is a plausible one since it means that labor demand, which depends negatively on the real wage, is not so sensitive in this respect that the wage sum is in fact decreased by an increase in the real wage. The mathematical condition underlying this assumption is that output elasticity with respect to real wages (in absolute terms) is less than 1 which is true at the steady state if the condition $(-d_1)\omega_e^s < \bar{u}y^p$ holds.

**Proof:** 1. Let us first consider the case where $\beta_p = 0, \beta_h = 0$ and $\kappa_p = 1$ hold and where therefore $\omega^s$ and $\pi_h$ stay fixed at their steady state values. The remaining 3D system in the state variables $l^e, k_h, d_w$ (in this order) then gives rise to a Jacobian $J$ at its interior steady state which is of the form

$$
J = \begin{pmatrix}
0 & 0 & + \\
0 & - & - \\
- & 0 & -
\end{pmatrix}
$$

23Loss of stability is not obvious for the parameter $\beta_h$, but is of the same type if it occurs.
if the parameter $\beta_w$ is chosen so small that $J_{33} < 0$ holds. It is again easy to show that the
Routh-Hurwitz conditions are fulfilled in such a case, in the same way as they were shown to
hold in proposition 2.

Let us next investigate the case where $\beta_p = 0, \beta_h > 0$ and $\kappa_p = 1$ holds where the resulting
dynamics therefore have become of dimension 4 (with $\pi_h$ as the fourth state variable). It is
then again easy to show that the determinant of the enlarged Jacobian can be reduced to the
form (if the assumption on $\beta_w$ is again made)

$$\det J = \begin{vmatrix} 0 & 0 & + & 0 \\ 0 & - & - & 0 \\ - & 0 & - & 0 \\ 0 & - & - & - \end{vmatrix}.$$ 

This determinant is therefore positive (since the upper $3 \times 3$ minor has been shown to be negative). Parameter values $\beta_h$ sufficiently close to zero therefore imply that the real parts
of the three eigenvalues which were negative (in the case $\beta_h = 0$) must stay negative also for
small positive $\beta_h$ which implies that the fourth eigenvalue will move from zero to a negative
value in order to allow for a positive determinant of the Jacobian of the 4D system.

We now move in the same way from $\beta_p = 0$ and $\kappa_p = 1$ to values of these parameters
sufficiently close to this situation. This the gives a 5D system whose fifth eigenvalue is no
longer zero by necessity. We show again that the determinant of the Jacobian of this 5D system
is negative and thus get in the same way as in the preceding step that the fifth eigenvalue
must change from zero to a negative value in order to fulfill the condition on the determinant
just stated. Therefore if the parameter changes are again that small that the negative real
parts of the first four eigenvalues remain negative we get in sum that all real parts of the
eigenvalues of the Jacobian of the full 5D dynamics must be negative, i.e., the interior steady
state is in fact locally asymptotically stable under the stated conditions (the proof has in fact
shown that there are at least three real eigenvalues in such a situation).

There remains to be shown that the determinant of the 5D Jacobian is indeed negative
under the stated conditions. To this end we first of all observe that the right hand side
equations of the dynamics (39) - (43) can be reduced to the following expressions on this
right hand side (in the case $\beta_p = 0$):

$$(\dot{k}_h) \ldots + \pi_h,$$

$$(\ddot{\pi}_h) \ldots - k_h,$$

$$(\ddot{l}^e) \ldots - g_k,$$

$$(\ddot{\omega}^e) \ldots + y/l^e,$$

$$(\ddot{d}_w) \ldots + y_w^D,$$

without change in the sign of the determinant to be investigated.

This gives rise to the following sign structure for this determinant:

$$\det J = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & - & - & 0 \\ ? & ? & 0 & + \\ ? & ? & - & - \\ ? & 0 & 0 & + \end{vmatrix} < 0,$$

which gives the desired result.
2. Assertion 2 is easy to show in the case $\kappa_p = 1$ since we then have that the parameter $\beta_w$ is only present in the fifth law of motion and there with a positive effect on the trace of $J$ via

$$J_{55} = \kappa \beta_w l^o_t (-d_2)/l^o_t d_w,$$

which means that the trace of $J$ can be made positive if $\beta_w$ is chosen sufficiently large. Note that things are more difficult in the case $\kappa_p < 1$ since we then have a stabilizing Rose effect of wage flexibility, which counteracts the destabilizing debt deflation effect of wage–price inflation (of the case $\kappa_p = 1$) just considered.

3. In order to prove this assertion we have to calculate that part of the considered determinant of the 5D system which depends on the parameter $\beta_p$. We again only show the items that are relevant for this calculation (where $\beta_p > 0$ now holds):

$$\det J(\beta_p) = \begin{vmatrix} 0 & + & 0 & 0 & 0 \\ - & 0 & 0 & + & - \\ 0 & 0 & 0 & + & + \\ ? & ? & ? & + & \beta_p \\ ? & ? & ? & + & ? \end{vmatrix}. $$

Here, $\det J(\beta_p)$ denotes only that part of the determinant which in fact depends on the parameter $\beta_p$. Inspecting the original 5D dynamics one of course notes that the parameter $\beta_p$ also appears in its fifth law of motion, but that it can be removed from the row of the corresponding Jacobian with respect to the calculation of their signs by means of an appropriate multiple of the fourth row without change in the qualitative structure of the remaining terms. Furthermore, it can be shown that $\beta_p$ can be removed from the fourth column of the above determinant (by means of $(-d_1/d_2)$ times the fifth column) without change of the plus sign in $J_{34}$.\footnote{See (50) and note in this regard that $\rho_x$ is given by $d_1 (1 - \omega^e l^o_t) - l^o_t \rho_y$ which implies that $J_{34}$ is larger than $J_{55}$ as was claimed above.} This implies as remaining terms for the considered determinant in its dependence on the parameter $\beta_p$:

$$\det J(\beta_p) = \begin{vmatrix} 0 & + & 0 & 0 & 0 \\ - & 0 & 0 & + & - \\ 0 & 0 & 0 & + & + \\ ? & ? & ? & + & \beta_p \\ ? & ? & ? & + & ? \end{vmatrix}. $$

We therefore get that the linear function $\det J(\beta_p)$ is upward sloping. Since we know already that $\det J(\beta_p)$ is negative for $\beta_p = 0$ we thus get that there is a unique value for $\beta_p$ where $\det J$ must be zero (and that it is positive thereafter).

Let us finally consider the full 6D dynamics (39) – (44) of this section and investigate to what extent monetary policy (44) can contribute to the stability of the 5D dynamics of the private sector. Due to the peculiar role of debt in the considered dynamics we however get a negative result in this regard:

**Proposition 7**

1. The interior steady state of the 6D dynamics (39) – (44), which in the 5D case was locally asymptotically stable for $\partial y^e_5/\partial \omega^e > 0$, $\beta_p, \beta_w, \beta_h$ sufficiently small and $\kappa_p$ sufficiently close to 1, becomes unstable (for all parameter choices) if the interest
rate policy rule is switched on by choosing a positive value for either $\beta_{r_2}$ or $\beta_{r_3}$ (the other parameters in this feedback policy being zero still).

2. Asymptotic stability is regained in the situation considered in assertion 1, if either $\beta_{r_2}$ or $\beta_{r_3}$ is negative and sufficiently small (the other being zero still).

3. In the situation considered in assertion 2, stability gets lost (in general) by way of a Hopf-bifurcation, if the parameter $\beta_p$ is made sufficiently large.

Proof: 1. The 6D dynamics (39) – (44) can now be reduced to the following form if attention is only given to the calculation of the sign of the determinant of the Jacobian at the steady state and if the case $\beta_{r_3} > 0$ is considered for example (the case $\beta_{r_2} > 0$ is proved in the same way):

\[
\begin{align*}
(\dot{t}^e) & \quad \ldots \quad -g_k, \\
(\dot{k}_h) & \quad \ldots \quad +g_h, \\
(\dot{\omega}^e) & \quad \ldots \quad +\beta_w de/l^e, \\
(\dot{\pi}_h) & \quad \ldots \quad +\beta_h\left(\frac{c^w}{k_h} - \bar{U}_h\right), \\
(\dot{d}_w) & \quad \ldots \quad +(c_1 + c_2 - 1)y^P_w - (g_k - \delta)d_w, \\
(\dot{r}) & \quad \ldots \quad +\beta_{r_3}\left(y/y^P - \bar{U}\right).
\end{align*}
\]

With the same objective in mind this situation can be reduced further to:

\[
\begin{align*}
(\dot{t}^e) & \quad \ldots \quad +\omega^e, \\
(\dot{k}_h) & \quad \ldots \quad +\pi_h, \\
(\dot{\omega}^e) & \quad \ldots \quad +1/l^e, \\
(\dot{\pi}_h) & \quad \ldots \quad +\beta_h\left(\frac{1}{k_h}\right), \\
(\dot{d}_w) & \quad \ldots \quad -r, \\
(\dot{r}) & \quad \ldots \quad -d_w.
\end{align*}
\]

It follows that the sign of $\det J$ must be negative which turns around one of the necessary and sufficient Routh-Hurwitz conditions for local asymptotic stability.

2. In the case assumed by assertion 2 we get, in the place of just shown result, that

\[
\begin{align*}
(\dot{t}^e) & \quad \ldots \quad +\omega^e, \\
(\dot{k}_h) & \quad \ldots \quad +\pi_h, \\
(\dot{\omega}^e) & \quad \ldots \quad +1/l^e, \\
(\dot{\pi}_h) & \quad \ldots \quad +\beta_h\left(\frac{1}{k_h}\right), \\
(\dot{d}_w) & \quad \ldots \quad -r, \\
(\dot{r}) & \quad \ldots \quad +d_w,
\end{align*}
\]

and thus $\det J > 0$ in this case. Continuity of eigenvalues with respect to parameter changes then again ensures that the stability result shown for the 5D case is preserved by such an addition of the interest rate policy rule.

3. Since $\det J$ is unambiguously positive in the situation considered by assertion 2 we immediately obtain the assertion from the fact that the trace of $J$ is an upward sloping linear
function of the parameter $\beta_p$, due to the destabilizing Rose-effect and the destabilizing Fisher-effect as far as price level flexibility is concerned and due to the fact that the $\beta_p$-term in the interest rate policy rule does not concern the trace of the matrix $J$. Note that we do not prove the (not very restrictive) speed condition of the Hopf-bifurcation theorem here (which in the present case is very difficult to obtain), but only assume that it will be fulfilled in nearly all conceivable situations.

Note that the seemingly perverse result of assertion 2 is not really implausible if one notes the following characteristic of the presently considered dynamics. A policy of decreasing nominal interest rates in the situation of a depressed economy (or a deflationary one) in order to push economic activity back to normal activity does not work well in the present context, since this tends to increase disposable income of workers and thus their consumption and indebtedness which by assumption leads to a further decline in the output of firms and thus does not necessarily have the consequences intended by this monetary policy (inducing further interest rate reductions). Monetary policy of this type therefore can only be expected to work if interest rate reductions speed up economic activity. Such a situation is however only present in the general 19D model of this paper, where investment behavior responds positively to a chain of interest rate reductions in general. This to some extent shows that the 6D dynamics investigated in this section must be embedded in not only the general 9D situation where sluggish quantity adjustments of Metzlerian type make the feedback chains on the market for goods less fast and more involved and where nominal price adjustments matter, but must allow for the case where long-term interest rates respond to short-term ones and thus lead to responses of investment behavior in view of the adjustments that occur in the financial markets. Such a task can however at present only be approached numerically, a few examples of which are discussed below (see appendix one for the corresponding figures).\(^{25}\) We stress that these illustrations are preliminary in nature and must be continued and extended in much more depth in future studies of this model type where also more extended debt deflation mechanisms than the still simple one of this paper should be integrated.

In figure 1 in the appendix 1 we show a case where damped oscillations are generated by the 9D dynamics in the case of a positive rental price shock, here still in the presence of a peg of the nominal rate of interest. We see that capacity utilization rates in the goods- and the labor-market are basically fluctuating in line with each other, while the capacity utilization rate of space is first leading and later on lagging behind these two measures of the business cycle (which then also become weaker in their positive correlation). These rates are all decreasing initially, since we had a positive rental price shock which not only reduces the demand for housing services, but also other consumption demand and thus economic activity. We have a less than normal return in the housing sector soon after the positive price shock in this sector due to a significant decrease in the demand for housing services, see the figure bottom right, which is accompanied by reduced capital formation in this sector relative to the goods-producing industry. This holds over a long-run horizon of fifty years, over which the demand for housing services does not return to its initial level again (though capacity utilization in the housing sector does reach high levels again). The opposite holds true, in particular with respect to the rate of return, of the goods manufacturing sector. Bottom right we finally see a mild cyclical evolution with respect to occupied rental space and rental prices. We stress that the considered situation is still an extreme one, since neither wages nor goods

\(^{25}\)The parameters underlying the numerical illustrations shown in appendix 1 are (up to the changes discussed with the figures there shown): $\beta_w = 0; \beta_p = 0; \beta_n = 0.3; \beta_n^d = 0.1; \beta_r = 1; \beta_r^o = 0; \beta_r^o = 0; \beta_r^o = 0; \alpha_{k_1} = 0.25; \alpha_{k_3} = 0.25; \alpha_{k_4} = 0; \alpha_{k_5} = 0; \kappa_p = 0; \kappa_p = 0; \theta = 0.92; \theta = 2; \theta = 0.9; \theta = 0.95; c_1 = 0.7; \tau = 0.08; \delta = 0.1; \tau = 0.5; \delta = 0.1; g = 0.1; \gamma = 0.06; c_2 = 0.4; y = 1; \kappa_h = 0; p_s = 5.$
prices respond to demand pressure on their respective markets so far, which allows for zero roots and thus path dependence and asymmetries in the time series shown. The considered situation is indeed a very sluggish one with respect to cycle lengths, since the economy does not respond to certain demand pressures to a sufficient degree.

In the figure 2 we have increased the adjustment speed of goods prices (away from its zero level to 0.2) which – due to an adverse real-wage or Rose-effect – destabilizes the economy leading to higher volatility in all variables just discussed. This also removes the path-dependency from the shown time series, allows for basically symmetric fluctuations of utilization rates around their steady state levels, with the rate of capacity utilization of space now always leading the other two measures of the business cycle. On average profitability in the housing sector remains still depressed, while the opposite still seems to hold in goods-manufacturing, implying that the capital stock underlying the supply of housing services is still shrinking relative to the one in goods-manufacturing. The variable that is subject to a positive shock is now the debt to capital ratio of workers households which leads to an immediate decline in their demand, in particular for housing services, a recession in all markets of the economy and the start of the business cycle from the resulting decrease in economic activity. There is now significant overshooting and a nearly persistent cycle in the interaction between rented space and rental prices and a pronounced negative correlation in the evolution of the rates of return in housing services and manufacturing.

Next, in figure 3, we allow for much stronger price adjustments, and now also adjustment of wages with respect to demand pressure on the labor market, and return to the case of a positive shock in rental prices. We now indeed get price deflation with respect to all three price levels of the model. We also allow now for an active interest rate policy of the central bank which here follows economic activity closely and is thus meant to be countercyclical. The negative correlation between the rates of return in the provision of goods and space is still there and there is positive correlation between our three measures of economic activity, which now exhibit a significant downward trend in addition. This is the novel thing in this now cyclically fairly explosive situation accompanied by the significant upward trend in workers’ debt to capital ratio and the shown downward trend in space rental prices as well as occupied space.

The explosive fluctuations of the preceding figure can however be removed and turned again into damped oscillations when wages, though remaining flexible in the upward direction, are made downwardly rigid by an appropriate nonlinearity in the employed money-wage Phillips curve.26 This is shown in figure 4 where the trends in the debt to capital ratio and the rental prices are removed by this downward rigidity in nominal wages (an important cost-pressure term in the evolution of space and goods prices). This asymmetric rigidity therefore helps to overcome the deflationary forces indicated in the preceding figure. Yet, due to the lack of a downward adjustment in the money wage we have no longer a uniquely determined NAIRU level on the labor market and need not have a situation in which the rate of employment recovers to its original steady state level (which is determined exogenously).

Figure 5 in this respect finally shows what indeed can happen in the economy if in particular this downward rigidity of money wages is removed to a larger degree. The shown situation of a strong process of debt deflation and increasing depression must however be considered in its details in much more depth than is possible here. In this paper we primarily attempted to supplement other work by the authors (and Willi Semmler) on the occurrence of debt deflation forces in the sector of firms by here considering debtor-creditor relationships in the

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26See Chiarella, Flaschel, Groh and Semmler (2000) for a detailed discussion of this type of downward rigidity in the money-wage Phillips curve.
household sector solely and the dynamics this particular relationship may imply for the long-run evolution of the macroeconomy.

5 Summary

In this paper we have reconsidered a general disequilibrium model of an applied orientation, exhibiting a detailed modelling of the private housing sector, which we have developed in a series of working papers starting from the Murphy model for the Australian economy. This modelling approach is complete with respect to budget equations and stock-flow interactions and can be reduced to a somewhat simplified 18D core model, the dynamics of which was intensively studied in this earlier work. In the present paper we have modified this type of model building towards the explicit consideration of debtor and creditor households which extended the dynamics of the core model by 1 dimension to 19D by the addition of the dynamics of the debt to capital ratio of the indebted households. Various subdynamics of these 19D dynamics were investigated theoretically and illustrated numerically. The basic findings were that there is convergence to the balanced growth path of the model for sluggish disequilibrium adjustment processes, that persistent investment cycles in the housing sector can be generated for certain higher adjustment speeds by way of Hopf-bifurcations in particular, and that processes of debt deflation may trigger monotonic depressions that get more and more severe when the real debt of debtor households is systematically increased by deflationary spirals in the goods manufacturing sector in particular.

6 References


7 Appendix 1: Graphs

In this appendix we briefly collect some numerical illustrations, see the close of section 4 for comments on them, of the investment cycles that are implied by this model and the processes of debt accumulation and debt deflation it can give rise to. These numerical illustration only provide a first impression of the dynamics that the model is capable of generating and thus here indeed only serve the purpose of illustrations. Detailed numerical simulations must follow later on which in particular should take a closer look on the feedback channels that characterize the dynamics of this paper.

Figure 1:
*Damped fluctuations in the supply of housing services and rental prices*
Figure 2:
Explosive fluctuations through flexible goods-price level adjustments

Figure 3:
Explosive fluctuations and debt deflation
Figure 5: Damped fluctuations based on absolute downward wave height.
8 Appendix 2: Notation

The following list of symbols contains only domestic variables and parameters. Foreign magnitudes are defined analogously and are indicated by an asterisk (*).

A. Statically or dynamically endogenous variables:

\[ Y \]
Output of the domestic good

\[ Y^d \]
Aggregate demand for the domestic good

\[ Y^p \]
Potential output of the domestic good

\[ Y^e \]
Expected sales for the domestic good

\[ Y^D_n \]
Nominal disposable income of workers

\[ Y^D_n \]
Nominal disposable income of asset holders

\[ L_1 \]
Population aged 16 – 65

\[ L_2 \]
Population aged 66 – ...

\[ L_0 \]
Population aged 0 – 14

\[ L^d \]
Total employment of the employed

\[ L^d_f \]
Total employment of the work force of firms

\[ L^g = L^w_g \]
Total government employment (= public work force)

\[ L^w_f \]
Work force of firms

\[ L^w \]
Total active work force

\[ V^w_f \]
Employment rate of those employed in the private sector

\[ V = L^d / L \]
Rate of employment (\( V \) the employment–complement of the NAIRU)

\[ C_w \]
Real goods consumption of workers

\[ C_c \]
Real goods consumption of asset owners

\[ C_h^d \]
Supply of dwelling services

\[ C_h^d \]
Demand for dwelling services

\[ S^m_a \]
Nominal savings of asset holders

\[ I \]
Gross business fixed investment

\[ I_h \]
Gross fixed housing investment

\[ T \]
Planned inventory investment

\[ N \]
Actual inventories

\[ N^d \]
Desired inventories

\[ r \]
Nominal short-term rate of interest (price of bonds \( p_b = 1 \))

\[ r_l \]
Nominal long-term rate of interest (price of bonds \( p_b = 1/r_l \))

\[ \pi_b = \pi_b^c \]
Expected appreciation in the price of long-term domestic bonds

\[ T^n \]
Nominal (real) taxes

\[ G \]
Real government expenditure

\[ \rho^e \]
Expected rate of profit of firms

\[ \rho^a \]
Actual rate of profit of firms

\[ \rho_h \]
Actual rate of return for housing services

\[ K \]
Capital stock

\[ K_h \]
Capital stock in the housing sector

\[ w^b \]
Nominal wages including payroll tax

\[ w \]
Nominal wages before taxes

\[ w^u \]
Unemployment benefit per unemployed

\[ w^r \]
Pension rate

\[ p_y \]
Price level of domestic goods

\[ p_x \]
Price level of export goods in domestic currency
All parameters that follow represent positive magnitudes.

$P_m$  
Price level of import goods in domestic currency including taxation

$P_h$  
Rent per unit of dwelling

$\pi^l = \hat{\pi}^c_y$  
Medium-run expected rate of inflation

$P_b = 1/r_l$  
Price of long-term bonds

$\pi_{hs} = \hat{\pi}_h^c$  
Expected rate of bond price inflation

$e$  
Exchange rate (units of domestic currency per unit of foreign currency: A$/$)

$\epsilon = \hat{\epsilon}^c$  
Expected rate of change of the exchange rate

$L$  
Labor supply

$B$  
Stock of domestic short-term bonds

$B_w$  
Short-term debt held by workers

$B_e$  
Short-term debt held by asset owners

$B^l$  
Stock of domestic long-term bonds, of which $B^l_1$ are held by domestic asset-holders and $B^l_1^*$ by foreigners

$B^l_2$  
Foreign bonds held by domestic asset-holders

$\tau_m$  
Tax rates on imported commodities

$X$  
Exports

$f^d$  
Imports

$\tau_w$  
Tax rate on wages, pensions and unemployment benefits

$g_k$  
Actual rate of growth of the capital stock $K$

$g_h$  
Actual rate of growth of the housing capital stock $K_h$

$c^g$  
Actual consumption of goods per unit of capital

$c^w$  
Actual consumption of housing services per unit of capital

$d_g$  
Actual public debt / output ratio

$D_f$  
Actual nominal debt (loans) of firms

$r_d$  
Interest rate on the ratio of firms

$d_f$  
Actual debt / output ratio of firms

$l_e^c$  
Actual labor intensity (subject to Harrod neutral technical change)

$l_e^{de}$  
Actual employment of firms per unit of capital

$l_e^{fe}$  
Actual labor force of firms per unit of capital

$l_e^{we}$  
Actual employment (including the government sector) per unit of capital

$l_g$  
Actual labor force of the government sector per unit of capital

**B. Parameters**

$\delta$  
Depreciation rate of the capital stock of firms

$\delta_h$  
Depreciation rate in the housing sector

$n$  
Natural growth rate of the labor force

$n_l$  
Rate of Harrod neutral technical change

$\alpha_l^2$  
All $\alpha$-expressions: behavioral or other parameters

$\alpha_s$  
Proportion of adaptively formed expectations

$\beta_x$  
All $\beta$-expressions: adjustment speeds

$\gamma$  
Steady growth rate in the rest of the world (here = $n + n_l$)

$V$  
NAIRU employment rate on the external labor market

$V_f^{we}$  
Normal employment rate of the employed

$\bar{U}$  
Normal rate of capacity utilization of firms

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27 All parameters that follow represent positive magnitudes.
\( U_h \)  
Normal rate of capacity utilization of the stock of houses

\( \kappa_{w, \kappa_p} \)  
Weights of short- and long-run inflation \((\kappa_w \kappa_p \neq 1)\)

\( \kappa \)  
\( \frac{1}{1 - \kappa_w \kappa_p} \)

\( y^p \)  
Output-capital ratio

\( x_y \)  
Export-output ratio

\( l_y^e \)  
Actual labor intensity (in efficiency units)

\( j_y \)  
Import-output ratio

\( d_g \)  
Desired public debt / output ratio

\( \tau_c \)  
Tax rates on profit, rent and interest \((\tau_c^* = \tau_c)\)

\( \tau_p \)  
Payroll tax

\( c_1 \)  
Propensity to consume goods (out of wages)

\( c_2 \)  
Propensity to consume housing services (out of wages)

C. Further notation

\( \dot{x} \)  
Time derivative of a variable \( x \)

\( \dot{\bar{x}} \)  
Growth rate of \( x \)

\( r_o, etc. \)  
Steady state values

\( y = Y/K, etc. \)  
Real variables in intensive form

\( D_f = D_f/(p_y K), etc. \)  
Nominal variables in intensive form

\( \nu = N/K \)  
Inventory-capital ratio