Follow the Leader
Steady State Analysis and Emergence in a Dynamic Social Network

David Goldbaum*
University of Technology Sydney†

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Abstract
A leader/follower hierarchy is shown to be the unique equilibrium structure for a social network of individuals seeking early adoption of a subsequently popular trend, a socially defined outcome. The environment is relevant to settings in which leadership and timing of decisions are important or where being perceived as a trend setter is rewarded, as in technology, fashion, politics, and fields with opinion experts. The environment necessitates the hierarchy. A dynamic model with evolving social links is employed to test for the emergence of the equilibrium structure.

Keywords: Dynamic Network, Social Interaction, Consumer Choice
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†School of Finance and Economics, PO Box 123 Broadway, NSW 2007 Australia, david.goldbaum@uts.edu.au
1. Introduction

Certain settings reward early adopters of a subsequently popular trend. Consider, for example, the financial reward to the real estate developer (as well as the home buyer) who invests in an area just prior to gentrification. The claim, “Wine geeks just love bragging rights. They get kudos from their peers when they get a high-score wine first or get it cheaper” suggests the social reward one might earn for discovery prior to the information becoming common knowledge. An art collector may be both financially and socially rewarded for the purchase of a work by an artist prior to gaining broader appreciation and appeal. A politically active public figure who meaningfully endorses and campaigns for a successful candidate early in the election cycle when the outcome remains in doubt may gain clout or influence that is neither financial nor social.

In each of these scenarios, an individual benefits from acting in advance of a socially determined phenomenon, the emergence of which may be influenced by the individual’s action. The model developed herein formalizes just such a reward environment, considers the impact on the social order, and the possible influence on produce adoption. What distinguishes this model from other examinations of socially determined rewards, such as the minority game based on Arthur’s (1994) El Farol problem and the Brock and Durlauf (2001) reward to conformity, is the inclusion of time, allowing for the endogenously determined dissemination of choice based on strategic behavior with regards to the emergence of the social phenomenon. Individuals decide when to act on a social phenomenon through selective employment of a social network. In so doing, the emergence of the phenomena is determined by optimizing decisions of the individuals within the population. The result is an endogenous network of social influences determining the social influences which drive the social phenomenon. In equilibrium, the trends, the trend setters, the population that adopts a choice making it into a trend, and the social network by which influence flows are all endogenously determined, reflecting the nature of the product, how it is to be enjoyed, and the informational needs of the population in advance of making a decision.

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1 Steven Bialek quoted in Haeger (1998, para 9).
2 As illustration, Thornton (2009) discusses the art collector David Teiger who is also an honorary trustee at the Museum of Modern Art in New York, “He enjoys being a player in the power game of art, particularly at this level where patronage can have an impact on public consciousness,” and quotes Teiger: “My goal is to acquire works that great museums letch after.” (p.100). Thornton (2009) also observes “Unlike other industries, where buyers are anonymous and interchangeable, here, artists’ reputations are enhanced or contaminated by the people who own their work.” (p.88)
on adoption.

The social sciences offer considerable evidence of social influences on individual decisions. Early findings of influence can be found in the Lazarsfeld et al (1948) examination of voting patterns, as well as in consumer behavior found in Whyte (1954), Katz and Lazarsfeld (1955) and the field experiment by Amdt (1967). More recent examples include Hill et al (2006), who make use of mobile phone networks, and Dwyer (2007) who observes the influence of online chat room discussion on product decisions. Social networks are an important tool for examining social influence, including the early examples of Schelling (1971) and (1973), Katz and Shapiro (1985), and Banerjee (1992), all of which model the bidirectional interaction between individual decisions and global behavior.3

The architecture of a social network defines how information and influence disseminate across a population, as emphasized by Ellison (1993). Brock and Durlauf (2001) is an example for which peer effects are global, allowing tractable analysis using the technique of mean-field interactions. Localized influences affect employment in the models of Montgomery (1991), Topa (2001), and Calvo-Armengol and Jackson (2004). Cowan and Jonard (2004) document the impact of local and global connectivity on overall knowledge across a population.

Modeling individuals as proactively searching for information is in contrast to the more traditional models in which individuals are passive recipients of information and influence.4 There is traditionally no need to differentiate between sources because information is *ex ante* of uniform quality.5 In Watts and Dodds (2007), for example, each individual employs a threshold rule, adopting an option when a sufficiently large number of peers are seen to have previously adopted, suggesting that the social gains outweigh any individual cost to adopting. A threshold rule offers the additional benefit of aggregating across peers, filtering the influence of idiosyncratic behavior on individual decisions, when present, if the individual gains to adoption are uncertain.

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4 Global peer models such as Brock and Durlauf (2001) exclude the possibility of differentiation by individual. Local peer effect models such as Ellison and Fudenberg (1995) and Cowan and Jonard (2004) consist of exogenously determined fixed peers with whom the individual interacts (possibly based on a random draw). Ellison (1993) is typical of evolutionary game theory models in that individuals update their game strategy based on realized experiences from random pairings, but they are not given the option to select with whom they play. Banerjee and Fudenberg (2004) does allow for bias sampling of the population and results based on biased reporting, but this still differs from the current project in that the sampling strategy is not optimizing strategic behavior.  
5 There can be found models that differentiate between “strong” local links and “weak” global links, as in the Goldenbert et al (2001) examination of a small worlds network on population level behavior.
To the contrary, evidence from the marketing literature suggests that individuals do actively seek selective information, discriminating between information sources.6 This behavior can arise as a result of issues concerning trust or due to heterogeneity in information quality. Heterogeneity is consistent with a pervasive theme in the word-of-mouth literature: the existence and identification of early adopters and opinion leaders who exert influences over the information dissemination and choices made by others. Examples include Feick et al (1986) and Feick and Price (1987). Discriminating information based on source is also consistent with the developed model’s derived equilibrium behavior.

Selective information gathering suggests that the relevant component of the network by which information and influence travels is endogenous to agent information choice. Jackson and Wolinsky (1996) provides the setting employed in a number of models examining endogenous network formation, including Bala and Goyal (2000), Dutta and Jackson (2000), Watts (2001), Jackson and Watts (2002), and with some modification, Kirman et al (2007). In these models, the individual payoff function is based on connectivity. The individual’s strategy is to form links that maximize connectivity at minimal cost. The payoff function reflects the notion that connectivity is good, without explicitly modeling the details of how connectivity translates into a benefit. In the current project, the individual’s objective is to establish a link that facilitates early adoption of a subsequently popular trend, an objective not necessarily served by maximizing connectivity. The reward is based on how the social phenomenon plays out on the network, not on the network directly. In the presence of a leader, optimal strategy is to differentiate between individuals based on the quality of the information they provide. Of the models on network formation, the decision choice faced by individuals in equilibrium is more closely captured by the model of D’Souza et al (2007) in which individuals, when given the choice, select a link close to the base of a network tree.

Individuals solicit information in order to reduce uncertainty. Uncertainty varies by product. It might be that the consumer knows what he or she wants from the product but is uncertain which of the various product models best conforms, as might be the case when purchasing a new electronic product such as a digital camera or a computer. In other cases, the consumer may not feel confident in his or her ability to discern what constitutes a good product,

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as may be the case for an inexperienced art or wine buyer. Yet another possibility might be that the individual is simply looking for guidance about what will be the popular choice, for example with clothing and fashion, where a good decision depends on the subjective opinions of others.

This last category of information is the primary focus of this examination. The information being sought is about a yet-to-be realized social phenomenon. The outcome of the social phenomenon is influenced by the process of information gathering. This is the complexity of the system under examination. The architecture of the social network determines the social outcome. The architecture is endogenous to the process of strategic information gathering. Which information sources are deemed important depends on the quality of their information. The quality of the information is socially determined and thus endogenous to the network. The developed model demonstrates that a leader/follower hierarchical structure is the natural endogenous product of the population’s desire for early adoption. What remains for the fully endogenous setting is a process for determining which of the possible equilibrium hierarchies prevails.

Ellison and Fudenberg (1995) and Banerjee and Fudenberg (2004) examine herding on a social network, distinguishing between processes that produce good learning that facilitates adoption of the better choice and inefficient herding that produces popularity independent of product quality. Quality is defined exogenously prior to the selection process and in the absence of choice externalities. Social links are thus employed in those models to gather information about the intrinsic quality of the good. It may be possible to distinguish restaurants by the quality of their food, but the value in going to a bar or club is defined socially.

Pesendorfer’s (1995) explanation that the early adoption of new fashion is motivated by an effort to better attract potential mates by signaling wealth through conspicuous spending relies on a price premium for newly introduced fashion that declines with the passage of time. The current model offers an alternate mechanism of individual gain through early adoption by generating different rates of adoption of competing simultaneously introduced fashion items. Demonstrating the ability to precede a trend signals an individual’s influence over social behavior or an ability to predict it. Both traits are desirable when seeking to attract a potential partner as they can also be employed for other forms of individual gain. The benefit of early

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7 A general model is introduced that will allow exploration of the other two information seeking scenarios in subsequent studies.
adoption is also present in the financial market model of Challet et al (2001), where a trading strategy is profitable if adopted before it becomes popular but not if adopted after. For Challet et al, popularity is a social phenomenon but not one driven by social interactions.

The dynamic model contains similarities with Conlisk et al (2001) which, after initial independent choice, allows individuals to imitate according to a social network of weighted links, possibly leading to a switch in choice from an unchanging set. The authors are interested in the long-run properties of population choice under a variety of environments and decision rules. The current paper makes endogenous the quality of the choice based on a social outcome and makes endogenous the social network so that it reflects individual experiences and observations with an interest in the long-run properties of the social structure that lead to choice.

This model offers an opportunity to consider the formal and informal institutions by which information disseminates through a population as endogenous to the product choice, how the product is to be enjoyed or used, and the information needs of the individuals choosing among the different options. Understanding the economic and social needs addressed by these institutions is important in an information driven economy and culture.

Conveniently, the timing problem faced by the individual in the population can be mapped into a spatial positioning problem captured by tiers on a social hierarchy. The ability to imitate someone in the time domain becomes the ability to link below the same someone in the spatial domain. As a spatial problem, individuals must decide whether to join a hierarchy and if so, which of the links available to them they want to employ. The analysis tackles the second problem first, developing an optimal strategy for choosing which link to employ when linking to the hierarchy. The characteristics of the resulting hierarchy are analyzed.

The option for each individual to act independent of a leader and her hierarchy of followers means that the hierarchy cannot be presumed as an equilibrium social structure. The analysis examines the conditions under which individuals want to join the hierarchy. Also considered is how those not joining a dominate hierarchy should organize. Of particular interest is whether coexisting multiple hierarchies can exist in equilibrium.

The steady state analysis on social hierarchies is conducted in support of a dynamic model. The steady-state analysis indicates describes the different equilibrium social. The steady-state analysis offers nothing by way of explaining how a leader might emerge from an initially unstructured social environment, but one a leader is present, it indicates how the population
should organize around the leader.

The paper proceeds with the presentation of the model and derivation of equilibrium behavior and resulting social hierarchical structure in Section 2. The model is presented as a game and thus includes an environment for agent interaction, strategy options, and payoffs. The social structure that emerges is the result of optimizing individual decisions given the environment. Alternative social structures are considered and excluded as equilibria in Section 3. A dynamic model is developed in Section 4 and explored in Section 5 to understand the process by which an individual might emerge as a leader of a hierarchy. The concluding Section 6 includes a discussion of the model and extensions.

2. Model

2.1 Choice setting

This section formally describes the environment and defines terms and outcomes. The environment consists of a population of individual agents linked directly or indirectly through a social network. They attempt to adopt from among a set of options, receiving a payoff that rewards early adoption of a popular choice. An equilibrium is a set of individual strategies that are individually optimal for each agent given the environment, including the strategies employed by the other members of the population. Equilibrium strategies generate an equilibrium social structure. Here, the equilibrium strategies have the individuals organize into a single hierarchy with a leader and a population of followers.

2.1.1 Choice environment

Consider a population of \( L \) agents. Each agent \( i \in \{1, 2, 3, ..., L\} \) chooses from among \( K \) available options, \( k \in \{a, b, c, ..., K\} \).\(^8\) The choice represents the action of adopting a product or behavior for the given period. Agent \( i \)'s choice is captured in notation by \( k_i \). The \( K \) options are new to the agents so that they have no prior experience with these particular options. The option labels do not convey to the agents any particular order that might allow coordination on, say, the first, last, or “average” option.\(^9\) Similarly, the agent labels are for exposition and do

\(^8\) An option can be a particular consumer product or brand within a category, but should be thought of more broadly as including behavior that goes beyond simple consumer products to include, for example, style of dress, grooming decisions, issue advocacy, or commuting method.

\(^9\) It could be that the labels are not common to the individuals or that the labels have no commonly recognized ordinal structure given the available options.
not convey a ranking to the other agents in the population.

Once agent $i$ makes the choice $k_i$, the act can be observed by a sub-population as determined by a network of links connecting individuals. From each agent emanates $d$ one-way (directed) non-redundant links to other agents of the population. Agent $j$’s link to agent $i$ allows agent $j$ to freely observe the choice made by agent $i$ at the time the choice is made. Information flows in only one direction on a link but the existence of a link from $j$ to $i$ does not prevent a link in the opposite direction that would allow agent $i$ to observe the choice of agent $j$. The $d$ agents to whom agent $j$ has a link will be referred to as agent $j$’s “contacts”.10 Those linking to agent $i$ are the observers of $i$.

Except where explicitly noted, the population is presumed to be fully connected. Formally, Assumption 1 The population is fully connected by the social network, such that there exists for all $i$ and $j$, $i \neq j$, a path of links by which $j$ can imitate $i$.

2.1.2 Strategies

A strategy is a process by which an individual arises at a decision on which of the $K$ options to choose.11 All traders face the same strategy set. There are two basic strategies, each of which requires an additional strategy-dependent step after its selection.

One strategy for agent $i$ is to act independently, selecting from the $K$ options without social input. Those acting independently choose $k_i$ as a random draw with probability $P_i^k$ indicating the probability that agent $i$ chooses $k$,

$$\Pr(k_i = k) = P_i^k$$

with $\sum_{k=1}^{K} P_i^k = 1$. This procedure will be motivated after explaining the reward mechanism.

The alternative strategy is to employ social cues based on the observed actions of one’s contacts as inputs into the selection process. If this strategy is employed, the second stage requires that the agent devise and implement a mechanism to map the social signals into a decision. Examples of possible strategies include a threshold criterion to choose the first option selected by $x \in \{1,\ldots,d\}$ of one’s contacts or a decision to imitate the action of a particular

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10 “Contact” is employed by Conlisk et al (2001) in the same context
11 Doing nothing and thereby not selecting one of the options is not consistent with the payoff structure developed below.
contact whenever that contact is observed to act. Once the environment and payoffs have been fully developed, the latter imitation mechanism is demonstrated consistent with the equilibrium.

2.1.3 Reward

Time is partitioned into discrete rounds, indexed \( r \in \{0, 1, 2, 3, \ldots, R\} \). Agents must settle on their strategy, whether to act independently or to employ social cues, in round 0. Also in round 0, those employing the social strategy must implement an adoption mechanism based on the observable actions of their contacts. All strategies are selected without communication and submitted simultaneously. Round 1 is the first opportunity to select one of the options.

The reward function captures the notion of a reward for early adoption of a popular choice. The agents possess von Neumann-Morganstern utility. The payoff is defined in terms of an action taken at a point in time relative to the actions of others. Let the utility payoff to agent \( i \) derived from the adoption of \( k \) depend on two socially defined components,

\[
\pi_i = J(N'_i) + T(N''_i) .
\]

To concentrate on the social influences, a possible third term offering individual reward independent of the social outcome, \( u_i(k) \), is left for subsequent analysis.

The \( J(N'_i) \) component captures the social interaction term, rewarding the agent for the popularity of a choice regardless of the round in which it is adopted. Here, \( N'_i \), represents the total number of agents to have chosen the particular option \( k \) the same as agent \( i \). The function \( J \) has \( J(x) \geq 0, J(1) = 0 \) and \( J'(x) \geq 0 \). Blume and Durlauf (2001) refers to this as the community effect, but thinking of \( J(N'_i) \) as the reward to “conformity” might better differentiate this component of the reward from that captured by \( T(N''_i) \).

The \( T(N''_i) \) element of the payoff rewards the agent for being an early adopter of a subsequently popular option. Here, \( N''_i \) represents the total number of agents who have adopted choice \( k \) that coincides with that of agent \( i \) in rounds subsequent to agent \( i \)’s adoption. Agent \( j \)’s inclusion in the \( N''_i \) of agent \( i \) does not require that agent \( j \) be an imitator of \( i \), either directly or indirectly through a chain of links, just that agent \( j \) subsequently makes the same decision as agent \( i \). Accordingly, the function \( T \) assigns a reward to agent \( i \) based on the number who adopt the same choice \( k \) in subsequent rounds with \( T(x) \geq 0, T(0) = 0 \), and \( T'(x) > 0 \).
The random draw for \( k_i \) is motivated by the absence of innate preferences concerning the individual options and the additional lack of identifying characteristics that might allow coordination leads each agent employing the strategy to act independently without social influence to chooses as though choosing randomly with \( p_i^k = 1/K \ \forall k, i \).

Delay is costly in terms of the lost opportunity to act before others in the population. It is thus optimal for those acting independently to make their choice in round 1 and for those employing social cues as inputs into their decision to adopt the choice dictated by their strategy as soon as the selection criteria has been met. For the latter group, criteria observed in round \( r \) allows the agent to act in round \( r + 1 \).

2.1.4 Properties of a hierarchy

While the payoff is defined in terms of an action taken at a point in time relative to the actions of others, there is a correspondence between the strategy induced time of a choice action and the individual’s position in a social hierarchy as determined by the links utilized by the strategic choices of everyone in the population.

**Definition 1** A hierarchy is a social structure consisting of a leader who acts independent of immediate social influences and a set of followers who imitate the action of the leader either through direct observation or indirectly via a chain of imitators originating with a direct link to the leader.

**Proposition 1** For every agent \( i \) in a fully connected social network there exists a hierarchy based on agent \( i \) as the leader and every agent \( j, j \neq i \), as a direct or indirect follower to agent \( i \).

**Proof:** Let Tier 0 of the hierarchy, the leadership position, be occupied by agent \( i \). Let Tier \( \delta \) of the hierarchy, \( \delta = \{1, 2, 3, \ldots, \Delta_i\} \) be populated by agents possessing a direct link to a member of tier \( \delta - 1 \). Tier \( \Delta_i \) is the final tier linking each of the \( j \) agents to agent \( i, j \neq i \). Assumption 1 of full connectivity ensures the existence of such a hierarchy for each agent \( i \).

Consider agent \( j \), a follower in a hierarchy led by agent \( i \). For \( d \geq 2 \), each agent \( j \) has a choice of how to link to agent \( i \). Agent \( j \)'s link-minimizing contact is the contact offering the most direct route to \( i \) involving the least number of links. The chain of links through which agent \( j \) imitates leader \( i \) is minimized if the number of links connecting \( j \) to \( i \) is the minimum number available from the universe of possible configurations that include \( i \) as the leader of a
hierarchy. Notice that to attain a minimized chain, not only does agent $j$ have to imitate the contact from her list offering the most direct route to the leader, but so too must each agent in the minimized chain. An minimized chain of links connecting follower $j$ to leader $i$ need not be unique in that there may be an alternate chain incorporating different members of the population but with the same number of links. The minimized chain of links places agent $j$ in the highest tier available to agent $j$ in a hierarchy under agent $i$'s leadership. All minimized chains place agent $j$ in the same tier of a hierarchy.

For $d \geq 2$, non-minimized chains of links exist for at least one members of the population. The first three frames of Figure 1 illustrates minimized and non-minimized chain of links possible under the leadership of agent 1 based on the contact lists $d_1 = \{3, 4\}$, $d_2 = \{1, 3\}$, $d_3 = \{1, 4\}$, and $d_4 = \{2, 3\}$. Agent 4 has two minimizing routes by which to imitate agent 1. Agent 2 is employing an non-minimized chain of links to imitate agent 1 if she does so indirectly, imitating agent 3.

**Figure 1**: Minimal and non-minimal hierarchies under agent 1

<table>
<thead>
<tr>
<th>Tier</th>
<th>Non-minimal Under 1</th>
<th>Minimal Under 1</th>
<th>Minimal Under 1</th>
<th>$H_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>3</td>
<td>2 3</td>
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<tr>
<td>2</td>
<td>2 4</td>
<td>4 4</td>
<td>4 4</td>
<td>4 4</td>
</tr>
</tbody>
</table>

**Definition 2** Minimal Hierarchy: A minimal hierarchy under the leadership of agent $i$ is one for which every agent connected to the hierarchy imitates $i$ through a minimized chain of links. Equivalently, an minimal hierarchy is one in which each agent in the hierarchy occupies the highest tier (lowest numbered tier) available to that agent given leader $i$. A hierarchy is thus minimal if the occupants of tier $\delta$ by way of a link to an occupant of tier $\delta - 1$ are without links to members of tier 0 through $\delta - 2$. A hierarchy will be non-minimal if there exists agent $j \neq i$ who could occupy tier $\delta_j$ but instead occupies tier $\delta_2$, $\delta_2 > \delta_j$ as a result of a failure by someone in the minimal chain to imitated their best positioned contact in the hierarchy.
Multiple versions of the minimal hierarchies exist if there exists an agent $j$ in tier $\delta$ with more than one contact in tier $\delta - 1$. If $A$ and $B$ are two different minimal hierarchies under the leadership of agent $i$, then if agent $j$ is in tier $\delta^A_j$ of hierarchy $A$ and $\delta^B_j$ of hierarchy $B$, then $\delta^A_j = \delta^B_j \forall j$. In this sense, all minimal hierarchies under the leadership of $i$ result in the same absolute tier position for each member of the population, though the links employed to create the hierarchy differ across minimal hierarchies. Let $H_i$ represent the minimal social hierarchy captured by the tier positions held by individuals in all minimal hierarchy under agent $i$. The fourth frame of Figure 1 illustrates for the previous example. The minimal social hierarchy $H_i$ is unique.

If the population forms into a single social hierarchy, then $N^T_i$ consists of the members of the population occupying tiers below agent $i$, that is, tiers numbered greater than $\delta_i$. In the event of multiple hierarchies or multiple individuals acting independently, agent $j$ of a different hierarchy than agent $i$ is counted as part of $N^T_i$ if $j$ occupies a tier below $i$ and if the two leaders chance to make the same choice so that $k_i = k_j$. The value of $N^T_i$ also includes the members of all hierarchies making the same choice as adopted by $i$.

Consider the payoffs received by position if there is full participation in a hierarchy in which agent $i$ as the leader. With full participation, $N^T_i = L$. Every agent in the population receives a conformity component of $J(L)$. For those agents occupying the lowest tier of the hierarchy $\pi_j = J(L)$ since, for these agents, $N^T_j = 0$ so that $T(N^T_j) = 0$. Those agents occupying tiers above the lowest tier, $\delta_j < \Delta_i$, receive a total payoff that increases with their distance from the lowest tier as followers are gained. At the top of the hierarchy, the leader receive the payoff $\pi_i = J(L) + T(L - 1)$.

2.2 Hierarchy as Equilibrium

For those choosing to incorporate social inputs into the selection process, choosing to imitate a single contact is demonstrated to be the equilibrium social strategy. Additionally, to link through the agent or one of the agents offering an link-minimizing route to the leader is an equilibrium strategy.
The minimal hierarchy is demonstrated to be the social structure resulting from a Nash equilibrium of individual behavior under appropriate conditions. This is accomplished by demonstrating that all individual followers prefer their position in the hierarchy to acting outside the hierarchy.

2.2.1 Minimal as Equilibrium

Further discussion is facilitated by labeling the different elements of the hierarchy $H_i$ according to the position relative to follower $j$. Let $m_{ij}$ be the number of followers in $H_i$ in tiers $\delta_j + 1$ through $\Delta_i$ and whose linkage to $i$ does not pass through $j$. Let $n_{ij}$ be the number of followers in $H_i$ who are occupants of tiers 1 through $\delta_j$, exclusive of $j$. Let $o_{ij}$ be those agents in $H_i$ who are linked to $i$ through imitation of $j$. In Figure 2, $n_{ij} = 1$, $m_{ij} = 4$ and $o_{ij} = 3$. The subscript “$ij$” is a reminder that the values for $m$, $n$, and $o$ are jointly dependent on $H_i$ as it exists under the particular agent $i$ and the position of agent $j$. Observe the identities

\begin{align}
N_i^j & \equiv 2 + m_{ij} + n_{ij} + o_{ij} \forall i, j, i \neq j \quad (2) \\
N_y^j & \equiv m_{ij} + o_{ij} \forall i, j \quad (3)
\end{align}

Figure 2: Positions in $H_i$ relative to $j$ by type

Let $N_y^j \mid dx_j$ represent the number $N_y^j$ for agent $j$ based on imitating contact $dx_j$ in a hierarchy led by agent $i$. Observe that $N_y^j$ is non-decreasing as agent $j$ moves up within a
hierarchy so that for contact \( d1 \) occupying tier \( \delta_1 - 1 \) and contact \( d2 \) occupying tier \( \delta_2 - 1 \) with \( \delta_1 < \delta_2 \), \( N_{ij}^T | d1_j \geq N_{ij}^T | d2_j \).

**Proposition 2** Given a leader \( i \), the optimal strategy for each follower includes the strategy that places agent \( j \) in her minimal position in \( H_i \)

**Proof**: For contact \( d1 \) occupying tier \( \delta_1 - 1 \) and contact \( d2 \) occupies tier \( \delta_2 - 1 \) with \( \delta_1 < \delta_2 \), \( N_{ij}^T | d1_j \geq N_{ij}^T | d2_j \). With \( T'(x) > 0 \), the contact offering the shortest route is at least weakly preferred to any other options.

In general, the relationship \( N_{ij}^T | d1_j \geq N_{ij}^T | d2_j \) will hold with strict inequality so that occupying the highest tier will be optimal for agent \( j \). Equality in \( N_{ij}^T | d1_j \geq N_{ij}^T | d2_j \) is a special case that arises for \( j \) if and only if \( m_{ij} = 0 \) for all minimal hierarchies under \( i \) (meaning that if agent \( j \) is not an occupant of \( \Delta_i \), she is the sole conduit through which those occupying the lower tiers can connect to \( i \)) and agent \( j \) has one of the other occupants of \( \delta_i \) as a contact. Because \( m_{ij} = 0 \) in all minimal hierarchies, the number of followers below \( j \) is \( o_{ij} \) regardless of whether she occupies tier \( \delta_i \) (minimally) or \( \delta_2 = \delta_i + 1 \) (non-minimally).

**Proposition 3** Given a leader \( i \),

a) \( N_{ij}^T | d1_j > N_{ij}^T | d2_j \) \( \forall j \) is a sufficient condition such that the optimal strategy for each follower is the strategy to imitate the contact offering the shortest route to the leader.

b) if \( \exists j \) such that \( N_{ij}^T | d1_j = N_{ij}^T | d2_j \) hierarchies offering the same payoff to \( j \) as the minimal hierarchy exist and can persist as an equilibrium social structure.

**Proof**: See appendix

[Comment on prop 3 here]

### 2.2.2 Conditions to join the hierarchy

Consider next the incentive for the potential follower to imitate a contact compared to singly acting independently. For agent \( j \), the payoff to conforming by remaining in \( H_i \) through the act of imitating is \( T(N_{ij}^T) + J(N_{ij}^T) \). Acting independently brings one of two possible outcomes. The preferred outcome for agent \( j \) is if she happens to select the same option as agent \( i \) so that \( k_j = k_i \). This occurs with probability \( 1/K \) and offers a payoff of \( T(N_{ij}^T - 1) + J(N_{ij}^T) \).
With probability \((K-1)/K\), agent \(j\) chooses differently than agent \(i\), receiving a payoff of \(T(N^T_j) + J(N^f_j)\). The necessary condition to keep agent \(j\) from deviating from hierarchical participation is

\[
T(N^T_j) + J(N^f_j) \geq \frac{1}{K} (T(N^T_i) - 1) + J(N^f_i)) + \frac{K-1}{K} (T(N^T_j) + J(N^f_j)). \tag{4}
\]

Using (2) and (3), (4) becomes Condition A:

**Condition A** \(T(L-2-n_j) + J(L) \geq \frac{1}{K} (T(L-2) + J(L)) + \frac{K-1}{K} (T(o_j) + J(o_j + 1))\)

**Proposition 4** Given a population of \(L \geq 3\) ex-ante homogeneous agents, differentiated only by their position in \(H\), if Condition A holds for follower agent \(j\) in \(H\), agent \(j\)’s optimal strategy is to conform by participating in the hierarchy given conforming behavior \(\forall j' \neq j\).

**Proof:** Follows directly from (4).

Further analysis is assisted by assigning functional forms to \(J\) and \(T\).\textsuperscript{12} Let

\[
J(N^f_j) = a_j (N^f_j - 1) \tag{5}
\]

\[
T(N^T_j) = a_r N^T_j. \tag{6}
\]

The coefficients \(a_r > 0\) and \(a_j \geq 0\) differentiate the payoff to agent \(j\) for being an early adopter from the payoff for making a popular choice. A value of \(a_j < a_r\) serves to down-weight the per-person conformity payoff relative to the per-follower early adoption payoff. The linear structure imposes constant marginal benefit for each additional member of the hierarchy and each additional follower.

Based on (5), (6) and (2), Condition A can be expressed as

\[
n_j (a_r - (K-1)a_j) \leq (a_r + a_j)m_j + (K-1)a_j. \tag{7}
\]

Since the right hand side of (7) is weakly positive, a sufficient condition for Condition A to hold for every follower in \(H\) is that \(a_r \leq (K-1)a_j\) or

\[
1 \leq (K-1) \frac{a_j}{a_r}. \tag{8}
\]

From (8), \(a_j > a_r\) ensures conformity for \(K > 1\). Choice options for which \(a_j < a_r\) introduce the possibility that “too few” \(K\) or “too little” \(a_j\) to support full participation in the

\textsuperscript{12} Once the model is developed based on linearity in (5) and (6) some deviations are considered.
hierarchy, as made precise by the analysis to follow.

In general, for \( j \in H_i \), \( 0 \leq n_j \leq L - 2 \) and \( m_j \geq 0 \). Further, \( m_j = 0 \) if \( \delta_j = \Delta_i \) and \( n_j = L - 2 \) iff \( \delta_j = \Delta_i \). Re-express (7) as

\[
(n_j + 1)(a_r - (K - 1)a_j) - m_j(a_r + a_j) \leq a_r
\]

and it becomes apparent that satisfying Condition A is “most difficult” for agent \( j \) in the lowest tier meaning that the set of \( a_r, a_j \), and \( K \) satisfying Condition A for \( j \) with \( \delta_j = \Delta_i \) is the smallest subset of the values satisfying Condition A for the general \( j \in H_i \). \(^{13}\) For occupants of the lowest tier, (9) becomes Condition B:

**Condition B**

\[
(1 - \frac{1}{L-1}) \leq (K - 1) \frac{a_j}{a_r}
\]

**Proposition 5** Given a population of \( L \geq 3 \), Condition B is a necessary and sufficient condition for conformity through imitation by every potential follower in \( H_i \).

**Proof:** Condition A holds for every follower \( j \in H_i \) if and only if Condition B holds. ■

If the conditions are such that the agent occupying the lowest tier to participate in the hierarchy, as is true if and only if Condition B holds, no other agent higher in the hierarchy will have incentive to deviate. Likewise, if Condition A does not hold for agent \( j \) in a middle tier, then there exists some individual below agent \( j \) in the hierarchy for whom Condition A also does not hold. This includes, but is not limited to, all individuals in tier \( \Delta_i \) and, if \( o_j > 0 \), all of agent \( j \)'s imitators (direct and indirect). This finding, that abandonment proceeds from the bottom of the hierarchy up, will be employed in subsequent analysis.

Condition B is independent of any \( i, j \), or \( H_i \) specific characteristics. This follows from the payoff in the lowest tier of the hierarchy that depends only on the size of the hierarchy, \( \pi_j = J(L) \), and not its structure for \( \delta_j = \Delta_i \forall H_i \).

Notice that as \( L \to \infty \) Condition B converges towards the sufficient condition expressed in (8).

The following are presumed to be true for subsequent analysis.

\(^{13}\) To illustrate, express (9) as \( a_j \geq a_j = a_r(m - n) / (m + (n + 1)(K - 1)) \). \( \partial a_j / \partial m < 0 \) and \( \partial a_j / \partial n > 0 \) for \( K > 1 \). \( a_r \) is the lower bound on \( a_r \) to ensure participation. The higher \( a_r \), the smaller the set of environment parameters that produce conformity.
**Assumption 2** Each \( H_i \) is evaluated based on the assumption that agent \( i \) is acting independently and that all other agents are aware that \( i \) is acting independently.

**Assumption 3** Agents \( j \neq i \), have full information regarding all of the possible states, including the position of each contact in \( H_i \).

**Proposition 6** Condition B is a necessary and sufficient condition for \( H_i \) to be a Nash equilibrium given leader \( i \) for all \( i \).

*Proof:* Consider play in which each agent faces the decision whether to act independently or to imitate. Imitating agents must decide which contact to imitate. Each \( H_i \) is a Nash Equilibrium in that given the actions of each of the other members in the population that produce \( H_i \), there is no individual incentive to deviate. There is no incentive for the leader, agent \( i \), to deviate and switch from choosing independently to imitation. In so doing, \( i \) would sacrifice all payoff as no independent choice would be made by anyone in the imitating population. Proposition 3 and Proposition 5 ensure that the hierarchy under \( i \) is the minimal hierarchy with full participation. Since Condition B is independent of \( i \), the condition holds for every agent in the population equally. ■

Consider the incentives captured by (7) and (9) that lead to deviations when Condition B does not hold. In both, a high value for \( (K - 1)a_j \) induces conformity to imitate. The benefit to acting independently is the possibility of choosing the same as agent \( i \), the leader, thereby gaining the payoff of the leader. A large \( K \) reduces the likelihood of this occurring and thus is a deterrent to independent action. The opportunity cost of abandoning the hierarchy is the certainty of the conformity payoff \( a_j(L - 1) \). The greater \( a_j \) relative to \( a_i \), the greater the incentive to conform.

Referring to Figure 2, for any agent \( j \), the incentive to deviate is based on the desire to gain \( n \) individuals as followers at the possible expense of losing the \( m \) followers. Those agents labeled \( m \) and \( o \) already follow \( j \) and will remain followers if \( k_i = k_j \). If \( j \) deviates and \( k_i \neq k_j \), agent \( j \) retains the population labeled \( o \) as followers, but sacrifices the followers \( m \) as well as conformity with \( i \), \( n \), and \( m \).

**3. Stability**

Should follower \( j \) act independently, it changes the payoff to every agent in the hierarchy
and possibly the optimality of the strategy they employed. This section examines three forms of stability: the size of the hierarchy when Condition B is not satisfied, the number of hierarchies that can exist in equilibrium, and the possibility that some hierarchies $H_i$ may exist as a Nash equilibrium but not as a subgame perfect equilibrium.

### 3.1 Partial participation

Given that should agent $j \in \Delta_i$ choose to act independently, this changes the payoff received by all of the remaining participants, altering each payoff element in Condition A, a concern, then, is whether a decision by agent $j$ to act independently causes a cascade of others in the hierarchy to choose to also act independently as well. At issue is the stability of the hierarchy. To evaluate, introduce the notion of fractional participation.

Let $\lambda$ be the proportion of the population that participates in the social hierarchy. Let $\Delta_i^\lambda$ represent the lowest occupied tier of a hierarchy based on $\lambda$ participation. Additionally, let $m_i^\lambda$, $n_i^\lambda$, and $o_i^\lambda$ represent appropriate positions in the remaining hierarchy, $H_i^\lambda$, consistent with Figure 2. Section 3.2 demonstrates that multiple hierarchies cannot exist if Condition B does not hold. For now, presume this to be the case so that the $(1-\lambda)L$ individuals not in $H_i^\lambda$ act independently.

A population of independent actors means that $N_k^I$ and $N_k^T$ are random variables for $k = "a", \ldots, K$. The relevant condition that determines whether agent $j$ occupying tier $\Delta_i^\lambda$ remains in $H_i^\lambda$ is based on Condition C.

**Condition C**

$$E(T(N_i^\lambda) + J(N_i^\lambda) \mid j \in H_i) \geq \frac{1}{K} E\left(T(N_i^\lambda) + J(N_i^\lambda) \mid k_j = k_i \right)$$

$$+ \frac{K-1}{K} E(J(N_k^{j \neq i}) \mid k_j \neq k_i)$$

Taking advantage of the linearity of the payoff functions, Condition C for agent $j$ can be evaluated based on the following expectations of participating and non-participating populations ($\lambda L$ includes agent $j$):
\[ E(N^T_{ki} \mid j \in H^\Delta_i) = \lambda L - 2 - n^\Delta_{ij} \]
\[ E(N^T_{ki} \mid j \in H^{\Delta^*}_i) = E(N^T_{ki} \mid k_j = k_i) = \lambda L + (1 - \lambda) L / K \]
\[ E(N^T_{ki} \mid k_j = k_i) = \lambda L - 2 \]
\[ E(N^T_{ki} \mid k_j \neq k_i) = \lambda L - 2 - n^\Delta_{ij} - m^\Delta_{ij} \]
\[ E(N^T_{ki} \mid k_j \neq k_i) = \lambda L - 1 - n^\Delta_{ij} - m^\Delta_{ij} + (1 - \lambda) L / K. \]

Use the expectations in (10) in (5) and (6) to express Condition C as:

\[ (n^\Delta_{ij} + 1)(a_r - (K - 1)a_r) - m^\Delta_{ij}(a_r + a_r) \leq a_r. \]

Observe that (11) has the same structure as (9). The position-dependent condition determining whether agent \( j \) in \( H^\Delta_i \) should conform or deviate is the same as for full participatory \( H_i \). If Condition C is satisfied for agent \( j \) occupying tier \( \Delta^*_i \), it is satisfied for all remaining followers in \( H^\Delta_i \). For agent \( j \) in tier \( \Delta^*_i \), \( m^\Delta_{ij} = 0 \) and \( n^\Delta_{ij} = \lambda L - 2 \). The necessary and sufficient condition for this agent to remain in the hierarchy is

\[ \left( 1 - \frac{1}{\lambda L - 1} \right) \leq (K - 1) \frac{a_r}{a_r}. \]

Let \( \bar{\lambda} \) be the value of \( \lambda \) for which (12) holds with equality,

\[ \bar{\lambda} = \frac{1}{L} \left( \frac{1}{1 - (K - 1) \frac{a_r}{a_r}} + 1 \right). \]

**Proposition 7** If Condition B does not hold, then given a fully connected social network with \( L \geq 2 \), there exists \( \lambda^* \),

\[ \lambda^* \in (\bar{\lambda} - 1 / L, \min(\bar{\lambda} + 1 / L, 1)), \]

with \( \frac{\delta}{L} \leq \bar{\lambda} \leq 1 \) as determined by (13) such that \( \lambda^* L \) is the equilibrium population size participating in the hierarchy. For \( \lambda L = \lambda^* L + 1 \), Condition A is violated for some agent \( j \in H^\Delta_i \). Further, the resulting hierarchy \( H^\Delta_i \) consists of the top tiers of \( H_i \) that contain \( \lambda^* L \) agents.

**Proof:** For \( (K - 1)a_r / a_r \in [0,1 - \frac{1}{(K - 1)}] \), \( \bar{\lambda} \) is a monotonically increasing function in \( (K - 1)a_r / a_r \) taking the extreme values of \( 2 / L \) and 1. Agent \( j \) with \( \delta_j = \Delta^*_i \) participates in the hierarchy if participation offers a higher expected payoff than non-participation. The range within \( \bar{\lambda} \pm 1 / L \) is
required to accommodate the discrete population in which agent \( j \)'s action affects the payoff options. If \( \lambda \geq \overline{\lambda} + 1/L \), then agent \( j \) with \( \delta_j = \Delta_j \) can unambiguously improve her expected payoff by acting independently. Likewise, for \( \lambda \leq \overline{\lambda} - 1/L \), agents acting independently with a link to a member of the hierarchy will be unambiguously better off joining the bottom of the hierarchy. Within this range, whether imitation or independent action offers higher profits is specific to \( H_j^* \). The hierarchy at its optimal size, \( H_j^* \), is thus stable with \( \lambda \) within the specified range of \( \overline{\lambda} \).

Consider the two scenarios that produce \( (K-1)a_j / a_f = 0 \). For \( a_j = 0 \) and \( K > 1 \), there is no conformity payoff and thus zero payoff to those agents in the lowest tier of the hierarchy. For \( a_f > 0 \) and \( a_j = 0 \) the only reward comes from having a following. If there is already a follower to a leader, then there is an incentive for the remaining population of non-leaders to act independently. There is no penalty to being the single follower, and thus indifference between following and acting independently when there are no other followers. This latter point explains the lower bound on \( \overline{\lambda} \) of \( 2/L \) rather than zero.

Similarly, with \( K = 1 \) and \( a_j > 0 \) there is no need to imitate in the absence of choice. Should there be a single follower, independent action dominates because of the certainty of gaining from another’s follower.

From (13), \( \overline{\lambda}L \), is constant with respect to \( L \). Thus, when Condition B does not hold, the number of participants in the hierarchy is fixed with respect to absolute size, not as a proportion of \( L \). With even moderate size \( L \), the initial drop-off in participation can be steep. As an example, for \( a_j / a_f = 0.1 / 1 = 0.1 \) and \( L = 101 \), then \( K = 11 \) satisfies Condition B, ensuring full participation but \( K = 10 \) violates Condition B with \( \overline{\lambda} = 0.11 \) so that only 11 agents participate in the hierarchy.

The interior equilibrium is stable in that deviations in \( \lambda \) from \( \lambda^* \) lead self-interested agents to join or abandon the hierarchy as appropriate to move \( \lambda \) towards \( \lambda^* \). This also eliminates autarky (everyone acting independently) as an equilibrium for \( \lambda^*L \geq 2 \).

### 3.2 Single and multiple hierarchies

The possibility that when Condition B is satisfied the population sorts into multiple
structured hierarchies, each with a different leader, has to be considered. Likewise, when Condition B fails to hold, the possibility that those who would otherwise act independently do not instead form into an alternate hierarchy under a different leader must also be examined.

Every position in a minimal hierarchy falls into one of the following situations: (1) leader, (who has $\delta_i = 0$); (2) a follower with at least one imitator (with $\delta_i \in [1, \Delta - 1]$); (3) a follower that acts before the last adopter, but without an imitator (also with $\delta_i \in [1, \Delta - 1]$); (4) those who are the last to adopt (with $\delta_i = \Delta$). Each is captured by the respectively numbered nodes in Figure 3.

Consider two hierarchies when Condition B is satisfied. Agent $i_A$ leads hierarchy $A$ and agent $i_B$ leads hierarchy $B$. Consider a process to affiliate the followers. All of the agents with a direct link to agent $i_A$ or agent $i_B$ occupy respectively tier $\delta_i = 1$, the top non-leader tier of hierarchy $A$, or tier $\delta_i = 1$, the top non-leader tier of hierarchy $B$. In general, the occupants of tier $\delta_h$, $h = A, B$ are able to link to an agent in tier $\delta_g - 1$ and are unable to link to any agents occupying tiers $1$ through $\delta_g - 2$, $g = A, B$. In contrast to the process of linking minimally under a single leader $i$, the presence of multiple equilibria means the link employed to occupy a given tier is important to the individual if the positions occupy the same tier of different hierarchies. Allow the population of such individuals to divide by some mechanism, the process of which does not affect the analysis to follow. The resulting hierarchies are jointly minimal in that each agent employs the shortest route to a leader.

Let $N_A^h$ be the number of participants in hierarchy $h$. For $N_A^h >> N_B^h$, hierarchy $B$ may be unsustainable as an equilibrium simply due to the dominate size of $A$. If $J(N_B^h) + T(N_B^h - 1) < J(N_A^h)$, then the lowest tier of $A$ offers a higher payoff than the leader position in $B$. The same inequality would have to hold for every member of $B$. Equivalently,
is sufficient to induce every member of $B$ with a link to $A$ to switch. Clearly $B$ cannot co-exist with $A$ as a Nash equilibrium in this scenario.

Consider more evenly sized hierarchies with, say, $N^J_B = N^J_A + 1$ such that $J(N^J_B) + T(1) > J(N^J_A + 1)$. An individual in $B$ with at least one follower prefers her current position to that of occupying the lowest tier of $A$, all else equal. This situation nonetheless cannot, in general, support the coexistence of $A$ and $B$ as an equilibrium. Let $A^*$ indicate $H_{i_j}$, the full participatory and minimal hierarchy of $i_j$. Let $A'$ and $B'$ represent the respective hierarchy after a switch in hierarchy association by agent $j$ from $B$ to $A$. Let $d_{ij}^A$ represent the number of links from agent $j$ pointing to $\{i: i \in A\}$ so that $d_{ij}^A > 0$ indicates that agent $j$ has at least one contact in hierarchy $A$.

Every feasible structure for $B$ generalizes to one of the following scenarios based on the hierarchy depicted in Figure 3 where each numbered position can represent an individual or a group. Cases I-IV are the full set of circumstances by which at least one agent in $B$ links to $A$ in compliance with the assumption of full connectivity.

**Case I:** A member of the lowest tier of $B$, as captured by agent 4 in Figure 3, has the option to link to $A$. Formally, this is expressed as $\delta_4^B = \Delta^B$ and $d_4^A > 0$.

**Case II:** The leader of $B$, as captured by agent 1 in Figure 3, has the option to link directly to $A$ and maintain her relative position to the followers in $B$. Formally, $\delta_1^B = 0$, $d_1^A > 0$, $d_2^A \geq 0$, $d_3^A \geq 0$, and $d_4^A \geq 0$ with the additional constraint that if any of the conditions on $d_2^A$, $d_3^A$, or $d_4^A$ hold with a strict inequality, then the existing links to $A$ place the agent in an inferior position to agent 1 so that $\delta_j^A \geq \delta_1^A + 1$, $j = 2, 3, 4$.

**Case III:** A sub-leader in $B$, as captured by agent 2 in Figure 3, has the opportunity to link to $A$ and the sub-leader’s fans are linked only to other members of $B$, not to $A$ directly. Formally, $d_1^A \geq 0$, $d_2^A > 0$, $d_3^A \geq 0$, and $d_4^A = 0$.

**Case IV:** An agent in $B$ with no direct imitators but occupying a tier above the lowest tier of $B$, as captured by agent 3 in Figure 3, has a link to $A$. Formally, $d_1^A \geq 0$, $d_2^A = 0$, $d_3^A > 0$, and

\[
\frac{N^J_B - 1}{N^J_A - 1} < \frac{a_j}{a_r + a_j}
\]

\[ (14) \]
Proposition 8 For two jointly minimal hierarchies, \( A \) and \( B \), with \( N_A' > N_B' \), in Cases I, II, and

\[ \exists j_B \in B \text{ with } d_j^B > 0 \text{ for whom } \pi_j^B > \pi_j^B. \]

Proof: For \( N_A' > N_B' \), in Case I,

\[ \pi_A^B = J(N_{A_4}^B) < J(N_{A_4}^A) \leq J(N_{A_4}^A) + T(N_{A_4}^{T_A}) = \pi_A^A \quad (15) \]

and thus it is beneficial for agent 4 to switch association from \( B \) to \( A \). In this case, \( N_A' = N_A^I + 1 \).

The \( J(N_{A_4}^B) \leq J(N_{A_4}^A) + T(N_{A_4}^{T_A}) \) component of (15) holds with equality if the link to \( A \) places agent 4 in the lowest tier of \( A \), \( \delta_{A_4}^A = \Delta_4^A \), so that \( T(N_{A_4}^{T_A}) = 0 \). The remaining hierarchy after the departure of agent 4, \( B' \), is characterized by Case I or Case II. The remaining defection from \( B' \) to \( A' \) potentially further benefit agent 4 since \( N_{A_4}^I > N_{A_4}^{I_4} \) and \( N_{A_4}^{I_4} \geq N_{A_4}^{I_4} \) so that

\[ \pi_A^B < \pi_A^{I_4} \leq \pi_A^{I_4}. \]

In Case II, agent 1 balances the certain benefits based on the position she holds in hierarchy \( A \) against the random payoff of acting independently as the leader of hierarchy \( B \). This decision was already examined in the derivation of Condition A. Under the assumption that Condition B holds, Condition A holds for agent 1. Agent 1 thus optimally rejoins hierarchy \( A \), forming \( H_A \).

In Case III, if agent 4 remains linked to agent 2, as required in a Nash equilibrium, then since \( N_{A_4}^I < N_{A_4}^I \), \( J(N_{A_4}^B) < J(N_{A_4}^A) \) and \( T(N_{B_2}^{T_B}) \leq T(N_{A_4}^{T_A}) \) so that

\[ \pi_A^B < \pi_A^{I_4}. \]

For \( N_A^I = N_B^I \), a switch by an agent consistent with the Cases I, II, and III generate improvement for the switching agent since the act of switching increases the size of the destination hierarchy above that of the originating hierarchy, \( N_A' > N_B' \) for a switch from \( B \) to \( A.\]

Even allowing agent 4 in Case III to consider her alternatives (as in a subgame perfect analysis), agent 4 would preferring to switch to \( A \) with agent 2 rather than remaining in hierarchy \( B \). Since Hierarchy \( B \) is minimal, agent 4 has no direct links to any occupants above \( \delta_{A_4}^B \), eliminating the option of linking to agent 1 directly. If a link exists that allows agent 4 to remain in \( B \), it is by linking to agent 3. Were agent 4 to link to agent 3, the new hierarchy would be captured by Case I, according to which the optimal choice is to switch to \( A \) by re-linking to 2. If the original \( B \) had \( d_i^A = d_j^A = 0 \), the assumption of full connectivity requires that one or both of
the remaining agents has a link to $A$ through either agent 2 or 4. Thus, the remaining hierarchy with just agents 1 and 3 is characterized by Case I or Case II. This offers further advantage to agent 2, who benefits by gaining position relative to agents 1 and 3. Thus, the remaining defection from $B'$ to $A'$ benefit agent 2, $\pi_2^B < \pi_2^{3''} \leq \pi_2^{3*}$.

**Proposition 9** For two jointly minimal hierarchies, $A$ and $B$, with $N^j_B \geq N^j_A$, in Case IV

$$j = 3 \in B$$ with $d^j_3 > 0$ and $\pi_3^{3''} > \pi_3^B$.

*Proof*: Under Case IV, the presence of agent 4 below agent 3 in $B$ makes it possible that $\pi_3^B > \pi_3^{3'}$. In the setting least favorable to defection, agent 3’s position in $A'$ places her in $\Delta_{4'}$, and $J(N^j_B) + T(N^j_B) > J(N^j_A)$. In this case, agent 3’s switch, all else equal, would reduce her payoff, making the coexistence of $A$ and $B$ a Nash equilibrium. We agent 3 to defect to $A$, by the assumption of full connectivity, at least one of the remaining agents has a connection to $A'$. The set of links that define Case IV ensure that any defections from $B'$ benefit agent 3. If linked to $A'$, agents 2 and 4 would have to link through agent 3 and agent 1 either links through agent 3 or links below 3. Thus $\pi_3^{3''} > \pi_3^B$. $\blacksquare$

**Proposition 10** For a fully connected social network under Condition B, the subgame perfect equilibrium social structure consists of a single minimal hierarchy, $H_i$. Multiple hierarchies cannot exist as a subgame perfect equilibrium. In Cases I, II, and III a single hierarchy under leader $i$ is also the unique Nash equilibrium. Multiple hierarchies can exist in a Nash equilibrium only in the presence of Case IV.

*Proof*: By Proposition 8 and Proposition 9, multiple hierarchies cannot exist as a Nash equilibrium under Case I, II, and III. In Case IV, multiple hierarchies can exist as a Nash equilibrium, but do not exist as a subgame perfect equilibrium. A defection by agent 3 to $A$ can result in $\pi_3^B > \pi_3^{3'}$. The remaining hierarchy $B'$, falls under Case I, II, or III, which cannot persist in equilibrium. Agent 3’s defection leads to more defections from $B$ that bring about the single hierarchy $A^*$. Since $\pi_3^B < \pi_3^{3''}$, agent 3’s subgame perfect strategy is to defect to $A$. $\blacksquare$

An example of Case IV is examined as an extensive form game in Appendix A.

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14 If the nodes in Figure 3 represent groups rather than individuals, it is possible that there are individual members of node 2 who, like agent 3, might have sufficient followers not directly linked to them such that a switch to join hierarchy $A$ reduces their payoff in a Nash evaluation. Such individuals are covered by the Case IV analysis.
demonstrating the existence of $B$ as a Nash equilibrium but not as a subgame perfect equilibrium. The key is that either agent 2 or 4 in Case IV would like to switch to $A$, but is unable to until a defection by agents 1 or 3 provides them a means to link to $A$.

Proposition 7 was proven based on the presumption that the $(1-\lambda)L$ non-participants all act independently. Consider the alternative, that when Condition B does not hold the $(1-\lambda)L$ individuals form an alternate hierarchy. The ability to form such a hierarchy is not assured by assuming full connectivity since it is possible that the chain of links connecting independent $j$ to independent $i$ passes through a member of $H^\lambda_A$. To consider an environment friendly to the development of an alternate hierarchy, presume a fully connected sub-population consisting of those individuals acting independently.

**Proposition 11** Suppose Condition B is not satisfied. Consider a fully connected sub-population of the agents \{ $j : j \notin H^\lambda_i$ \}. The equilibrium social structure consists of a single minimal hierarchy, $H^\lambda_i$. Those agents not in $H^\lambda_i$ act independently.

**Proof:** Consider the alternative. Assume two jointly minimal hierarchies $A$ and $B$. Condition C holds for both hierarchies so that $\lambda^A, \lambda^B \leq \lambda^*$ with at least one hierarchy holding at equality. If true for only one of the two hierarchies, then label that hierarchy $A$. Thus, $N^B_j \leq N^A_j$. If $N^B_j < N^A_j$, then joint efficiency in $A$ and $B$ is not an equilibrium structure. Any agent $j \in B$ with a link to an interior tier within $A$, $\delta_{Aj} < \delta_{Bj} < \Delta^\lambda_A$, will benefit from a switch to $A$ in accordance with Proposition 10. Once $\lambda^A = \lambda^*$, additional defections from $B$ to $A$ displaces agents from the lowest tier of $A$ equal to the number of switching agents from $B$. Allow the displaced agents from $A$ to join $B$. The result is two sequentially minimal hierarchies in which $H^\lambda_A$ is minimal with respect to the entire population up to size $\lambda^*L$ and $B$ is minimal with respect to the $(1-\lambda^*)L$ agents not in $H^\lambda_A$. Label this $H^{1-\lambda}_{B}$.

Consider agent $j$ occupying the lowest tier of $H^{1-\lambda}_B$, $\Delta_B$. Allow notation $m^A_{Bj}$ to represent the number of agents in $A$ for whom $\delta_{Aj} > \Delta_B$ so that they are in hierarchy $A$ but below agent $j$. $m^A_{Bj} = 0$ for $\Delta_A \leq \Delta_B$. Since agent $j$ is unable to profitably join $H^\lambda_A$ her two options are to remain in $B$ or act independently. To remain in $B$ requires $E(\pi^B_j) \geq E(\pi^A_j)$, where the “$I$” indicates acting independently. This reduces to Condition D.
Condition D \[ \left( 1 + \frac{N_d^J - 2 - m_{bij}^A}{N_B^J - 1} \right) \leq (K - 1) \frac{a_j}{a_r} \]

As a follower, the highest tier \( j \) can occupy in \( B \) occurs when \( \Delta_B = 1 \). In hierarchy \( A \), tiers 0 and 1 have to have at least one member each, so that \( m_{bij}^A \leq N_d^J - 2 \) ensuring that the left side of Condition D is weakly greater than one. For Condition D to hold, Condition B would have to hold as well (even stronger, the sufficient condition in (8) would have to hold), in violation of the initial assumptions of Proposition 11.

In the event that \( N_B^J = N_d^J \), \( A \) and \( B \) can remain jointly minimal as there is no size benefit to switching. Condition D is still violated, so that it would be beneficial for one of the two hierarchies to disband and act independently, but which should do so is arbitrary.\(^{15} \)

The condition that induces agent \( j \) to join an alternate hierarchy \( B \) is sufficient to ensure full participation in a single hierarchy. If Condition B does not hold, then Condition D is also violated and independent action is preferred to hierarchy \( B \). Hierarchies \( A \) and \( B \) thus cannot coexist.

### 3.2.1 Alternative environments

Condition D can be applied to consider whether an alternate hierarchy forms in the absence of full connectivity. Suppose Condition B holds, but that there are individuals in the population unable to link directly or indirectly to the current leader, \( i \). These individuals have the option to form an alternate hierarchy or act independently. An equilibrium alternate hierarchy can exist, but the minimum necessary and sufficient condition for participation in an alternate hierarchy \( B \) (evaluated for \( N_B^J > 1 \) since \( N_B^J = 1 \) is the equivalent to acting independently) requires a combination of \( a_r, a_j, K \) less favorable to independent action than is Condition B,

\[
\left( 1 - \frac{1}{L - 1} \right) < \left( 1 + \frac{(N_d^J - 2 - m_{bij}^A)}{(N_B^J - 1)} \right) \leq (K - 1) \frac{a_j}{a_r}.
\]

Another alternative to consider is the incorporation of a minority game component, similar to Arthur (1994), as a penalty for excessive popularity. This can be implemented by redefining

\(^{15}\) Those in the lowest tier of their hierarchy are indifferent and it is their action which determines which hierarchy disbands. Those in the upper echelon of their hierarchy would rather their hierarchy persist, but the persistence is not in their hands.
\[ J(N_i^j), \text{ for example, to} \]

\[ J(N_i^j) = \begin{cases} 
    a_j (N_i^j - 1) & \text{for } N_i^j < \lambda^* L \\
    0 & \text{otherwise}
\end{cases} \quad (17) \]

with \( 2 < \lambda^* L < L \). The result is an environment that supports a hierarchy of the size \( N_i^j < \lambda^* L \).

Again, those not in the main hierarchy may act independently or join to form an alternate hierarchy. Such a model imposes a penalty for excessive conformity should \( k_A = k_B \), though the reward to early adoption remains for those with followers. One can back out the condition for the formation of a hierarchy \( B \) with non-participants to either \( A \) or \( B \) acting independently.\(^\text{16}\) The equilibrium size of the alternate hierarchy can be backed out of (16) evaluated at equality on the right hand side by solving for \( N_B^* \) given \( a_i, a_j, \) and \( K \). If \( N_B^* < \lambda^* L \), the remainder act independently.

More generally, a concave conformity function with \( J(1) = 0, J'(1) > 0, J''(x) < 0 \) might also seem a reasonable alternative to the presumed linear function. This could be accomplished, for example, by replacing the constant \( a_j \) with a function \( a_j = a_j(N_i^j) \) for which \( a_j''(N_i^j) < 0 \).

In this case, Condition B is basically unchanged since it relies on only those in the lowest tier. It retains the same structure, substituting \( a_j(L) \) for \( a_j \). Without linearity, computing expected payoffs for fractional participation becomes considerably more laborious. Nonetheless, inference can be made based on the linear analysis. Concavity in \( J \) decreases the incentive for conformity with large populations while maintaining an incentive to conform in small groups.

To overcome the single hierarchy equilibrium requires that \( J(N_A^j) < J(N_B^j) \) for \( N_A^j > N_B^j \). A monotonically increasing concave function that retains \( J'(x) > 0 \) still produces a single hierarchy equilibrium. Generating multiple hierarchies in equilibrium requires \( J'(x) \leq 0 \) for large \( x \). A function with \( J(1) = 0, J'(1) > 0 \) and \( J(x) \to 0 \) as \( x \to L \) would offer higher conformity payoff for smaller hierarchies, inducing those who would be in the lowest tier(s) to abandon a large \( H_j \) in favor of forming alternative hierarchies. Such alternate hierarchies would still be subject to offer a conformity payoff greater than the expected payoff to acting independently as a condition for existence.

\(^{16}\) The participation condition in the lowest tier of \( B \) is not neat, but can be satisfied with a sufficiently large \( a_j \).
3.3 Hierarchies as subgame perfect

Every individual in the population is the leader of an equilibrium social hierarchy in which deviations by individual followers result in an inferior payoff to the deviant. Yet, unless the initial social structure was symmetric, each hierarchy is different. Some are more robust with regards to the payoff of the remaining participants than others. While every individual is the leader of a Nash equilibrium defined hierarchy, under special conditions, some of the same hierarchies cannot be supported as a subgame perfect equilibrium.

<table>
<thead>
<tr>
<th>Agents</th>
<th>Contacts</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>n o1</td>
</tr>
<tr>
<td>j</td>
<td>i o2</td>
</tr>
<tr>
<td>n</td>
<td>i j</td>
</tr>
<tr>
<td>o1</td>
<td>j o2</td>
</tr>
<tr>
<td>o2</td>
<td>j o1</td>
</tr>
</tbody>
</table>

Consider the social network captured by Table 1. Each member of the network has two contacts. Figure 4 depicts $H_i$ and $H_j$, both Nash equilibria. Consider the extensive form game in which each agent decides on a strategy in ordered play according to $i, j, n, o_1,$ and $o_2$. Evaluated thus, $H_j$ rather than $H_i$ is the subgame perfect equilibrium despite $i$’s ability to position herself as a leader in the first move. The full extensive form game is included in Appendix B. To summarize, a defection by $j$ to act independently attracts $n$ to link to the larger hierarchy offered by $j$ rather than to maintain the link to $i$. This leaves agent $i$ choosing between acting independently or joining $H_j$, the latter being the preferred choice of the two.

Figure 4: Alternate hierarchies under different leaders based on the links of Table 1

Two criteria are essential to undermine agent $i$’s leadership in a subgame perfect evaluation. The first is that there is a follower whose defection produces a hierarchy larger than what remains of $H_i$. The second criterion is that the potential defector must be motivated to
defect given the previously made decision by agent \( i \) to act independently. The presence of agent \( n \) offers agent \( j \) the opportunity for improvement. In \( H_i \), \( n \) chooses \( k \) contemporaneous with \( j \) and thus is not included in \( N^T_j \), but a defection by \( j \) to act independently draws agent \( n \) in as a follower of \( j \), a gain that exceeds the potential loss of agent \( i \) in \( N^T_j \). Were \( n \) not present in the population, then \( j \)'s deviation lowers her payoff given the previously made decision by \( i \) to act independently and thus would not be an optimal move. The two criteria are unlikely to arise in large populations unless generated by design.

4 Emergence

The static model has \( L \) Nash equilibria, one for each \( H_i \). For each agent, each of the different strategy choices is optimal in at least one of the possible equilibria. There is no mechanism within the static model environment that allows coordination on a particular leader without prior designation. In the absence of some indicator of who in the population is the leader, there is no dominant strategy.

The dynamic setting replicates this initially uncertain environment, starting from an unstructured social network without a designated leader. From the onset no particular strategy dominates. From a process of observation and adjustment emerges a leader and the minimal hierarchy, \( H_i \), based on the emerged leader.

The previous sections offer the conditions for existence of a hierarchy under the leadership of agent \( i \), describe what it means for the hierarchy to be minimal, and prove that the minimal hierarchy is unique and the only hierarchy that can exist as a subgame perfect equilibrium in the presence of a fully connected population. What has been left to this section is a process by which agent \( i \) in the population can emerge as the leader, a process that is examined using simulation.

4.1 Evolution in a Dynamic System

Looking to understand the emergence of a leader from an unstructured social network over repeated play with reasonable limits on information, many of the assumptions of the static model are relaxed. In the dynamic version, the population faces a new set of choices each period. As

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17 To see this, recognize that for agent \( i \) acting independent is optimal if the agent is the leader. Imitating contact \( d_i = j \) for all contacts is optimal for agent \( i \) if agent \( j \) is the leader.
with the static model, each agent establishes a strategy of independent action or imitation before
the first round of the period. Instead of engaging in rational strategic behavior, agents choose
among the available strategies probabilistically, updating these probabilities over time based on
individual observations and experiences.

Each period starts with each agent choosing whether to act independently or through
imitation. With probability \( \theta_{i,t} \), \( 0 \leq \theta_{i,t} \leq 1 \), agent \( i \) chooses to act independently.

With probability \( 1 - \theta_{i,t} \), agent \( i \) chooses to imitate one of the \( d \) contacts. Associated with
each of agent \( i \)'s \( d \) links is a weight, \( w_{i,j}^t \). The weight \( w_{i,j}^t \) is the probability that agent \( i \) imitates
agent \( j \), conditional on having chosen to imitate, so that

\[
\Pr(\text{agent } i \text{ imitates agent } j) = (1 - \theta_{i,t}) w_{i,j}^t \tag{18}
\]

with \( \sum_j w_{i,j}^t = 1 \).

As in the static model, the strategies selected by the agents at the start of the period are
played out over \( R \) rounds within the period. Introduce notation \( x^r \) to indicate the value of \( x \) in
round \( r \) of the period. In the final round, any agents yet to make a choice consider the population
and choose option \( k \) with a probability \( p_{k,t}^R \) that depends on the popularity of the option observed
in the previous round,

\[
\Pr(k_{i,t}^R = k | k_{i,t}^{R-1} \text{ unassigned}) = \frac{\exp(\rho N_{k,t}^{R-1})}{\sum_k \exp(\rho N_{k,t}^{R-1})}. \tag{19}
\]

At the end of each period, the agents know \( N_{k,t}^r \) for each \( k \) in each \( r \). The agents also know
the choice and round of adoption of each contact. The agents do not have knowledge of the full
network of links, neither those employed or those unemployed.

After the last round of the period, each agent adjusts the probability, \( \theta_{i,t} \), and weights, \( w_{i,j}^t \),
based on the perception of how successful each option would have been during the just
completed period. For the strategy actually employed, the agent knows the actual payoff
generated by the strategy. For the counterfactual strategies, the agents use the information they
have to approximate a speculative payoff.

With the knowledge of each contact’s choice and round of adoption, the agent uses existing
product and round adoption payoff information to compute the payoff they would have received

30
had they imitated the contact. This is not necessarily an accurate estimate since it ignores the
impact their own switching would have had on their own imitator’s choice, but without
knowledge of who her imitators might be, the agent does not possess the information to correctly
account for this. For each contact, the calculation is thus\(^{18}\)

\[
\tilde{N}_{d,t} = \sum_{r=t-1}^{B} N'_{r,d,t} \quad \text{and} \quad \tilde{N}_{d,t} = \sum_{r=1}^{R} N'_{r,d,t} .
\]

which are plugged into the payoff function to compute the speculative counterfactual payoffs.

Similarly, if the agents imitated, they need to speculate what the payoff would have been to
acting independently. Each agent has chosen one of the options as the one they would have
adopted had they acted independently. Naively, they take the payoff of having chosen that
option in the first period as their speculative payoff. This can substantial for those that would
have chosen a subsequently popular choice for that period. Let \( z'_{i,t} \) represent the payoff earned
by agent \( i \) by imitating agent \( j \) in period \( t \), whether actual or speculative.

The probabilities \( \theta_{i,t} \) and \( w'_{i,t} \) are adjusted according to an algorithm based on observation
and experience. Let \( A'_{i,t} \) be a cumulative performance measure associated with agent \( i \)’s
imitation of agent \( j \). A separate performance measure is maintained for each contact of agent \( i \).
The value of \( A'_{i,t} \) is updated according to

\[
A'_{i,t} = A'_{i,t-1} + (\pi_{i}(z'_{i,t}) - A'_{i,t-1})/t .
\]

Briefly, \( \pi_{i}(z'_{i,t}) \) is the period \( t \) realization of performance associated with agent \( i \) selecting to
imitate agent \( j \). The performance measure \( A'_{i,t} \) is simply the expected performance as determined
by a least-squares learning process as described in Marcet and Sargent (1989).

An asymptotically stable hierarchy requires that the randomness in the strategy decision
process converges to zero. There are a number of methods available to accomplish this. The
approach taken here is to employ a Replicator Dynamic (RD) process, by which higher
performing contacts attract greater weight at the expense of poorly performing contacts. Let \( \overline{A}_{i,j} \)
be the average performance across agent \( i \)’s contacts, weighted by current link weights.

\(^{18}\) If imitating a particular contact changes the individual’s choice or round of adoption, she also need to
accommodate the totals taking into account her own movement.
\[ \bar{A}_{it}^w = \sum_{j} w_{ij} A_{ij} \]  
Further, let a contact’s performance fall into the sets “good” or “bad” according to  
\[ \hat{d} = \{1, \ldots, d\} \],  
\[ Gd = \{ j \in \hat{d} \mid A_{ij} \geq \bar{A}_{ij}^w \} \], and  
\[ Bd = \{ j \in \hat{d} \mid A_{ij} < \bar{A}_{ij}^w \} \].

The weights evolved according to the process  
\[ w_{it} = w_{it}^j + \begin{cases} r(A_{ij} - \bar{A}_{ij}^w)w_{ij} & \text{for } j \in Bd \\ x \sum_{j \in \hat{d}} -r(A_{ij} - \bar{A}_{ij}^w)w_{ij} & \text{for } j \in Gd \end{cases} \]  
with  
\[ x_t = \frac{\zeta / |Gd| + A_t^j - \bar{A}_{ij}^w}{\zeta + \sum_{j \in \hat{d}} (A_{ij} - \bar{A}_{ij}^w)} \]
and  \[ r(x) = \tanh(\lambda_x x / 2). \]

This is a modification from the Branch and McGough (2008) K-choice RD process where the weighted average, \( \bar{A}_{ij}^w \), replaces their recommended simple equal weighted average performance. Without this modification, there is no mechanism for the highest performing contact to attract weight from other above average “good” contacts (particularly once all of the “bad” contacts reach zero weight).^{19}

Payoffs are normalized by the highest payoff contact,  
\[ \pi(z_{ij}^j) = z_{ij}^j / \max_k (z_{ij}^k). \]  
Similarly, \( \theta_{it} \) is updated according to  
\[ B_{it}^j = B_{it-1}^j + (\pi_B(y_{it}^j) - B_{it-1}^j) / t, \]  
\[ \theta_{it+1} = \theta_{it} + \begin{cases} \tanh(\lambda_y (B_{it}^0 - B_{it}^j)) / 2 (1 - \theta_{it}) & \text{for } B_{it}^0 \geq B_{it}^j \\ \tanh(\lambda_y (B_{it}^0 - B_{it}^j)) / 2 \theta_{it} & \text{for } B_{it}^0 < B_{it}^j \end{cases} \]  
\[ j = 0, 1. \] This process is simply the K-choice RD model in the two choice setting. The payoff \( y_{it}^0 \)

^{19} The process when employing weighted averaging converges to place all weight on the highest performing contact. If that process converges before relative position in the hierarchy stabilizes, the agent could settle on a contact that is eventually realized to be suboptimal. It might be natural to leave such a situation be, but a mechanism to allow the agent to adjust weights in such a case is introduced. The agent can check for the event \( \max(w_{ij}) = 1 \) and \( \max(A_{ij}) > \bar{A}_{ij}^w \), in which case the conditions for “good” and “bad” become  
\[ Gd = \{ j \in \hat{d} \mid A_{ij} > \bar{A}_{ij}^w \}, \quad Bd = \{ j \in \hat{d} \mid A_{ij} < \bar{A}_{ij}^w \} \]
for as long as the event remains true.
is the payoff to acting independently, \( y_{i,t}^0 = z_{i,t}^{\text{ind}} \), while the payoff \( y_{i,t}^1 \) is the payoff to imitating. The latter is taken by the agent to be the best payoff offered by one of her contacts, \( y_{i,t}^1 = z_{i,t}^{\text{max}} = \max(z_{i,t}^h) \).

### 4.3. Measuring the social network

The efficiency of a hierarchy is measured based on the cumulative distance of individual agents from their minimal tier. Let \( \delta_{i,t} \) represent the tier occupied by agent \( i \) in period \( t \). Let \( \delta_{i,t}^* \) indicate the minimal tier based on the current leader at time \( t \). Efficiency, \( E \), is thus measured as

\[
E = \sum_i \delta_{i,t}^* - \delta_{i,t}
\]

with \( E = 0 \) representing a fully minimal hierarchy while values of \( E < 0 \) represent deviations from efficiency.

### 5. Simulations

Each simulation starts with a population of individuals linked through a social network. The baseline initiation is to randomly assign the links, though symmetrically structured social networks were examined as well. The baseline is to assign the initial weights equally, \( \theta_{i,0} = 1/d \) and \( \theta_{i,0} = 0.5 \). The other baseline parameters of the simulation are reported in Table 2.

Despite the fact that no individual strategy can assure personal emergence as a leader, that some agent from the population does emerge as leader is quite robust in the developed environment. An example of the process of the time-series generated by the emergence is plotted in Figure 5. The curve labeled \( N(\text{ind}) \) is the number of agents choosing to act independently, \( \text{MaxFans} \) is the number of imitators of the agent with the highest number of fans that period, and \( N(r=\text{tr}) \) is the number of agents who make their selection in the final round. Also included is the measure of efficiency.
Table 2: Exogenous parameter settings used in the simulations

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Setting</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>100</td>
<td>Number of agents</td>
</tr>
<tr>
<td>$P$</td>
<td>500</td>
<td>Number of periods</td>
</tr>
<tr>
<td>$R$</td>
<td>20</td>
<td>Rounds per period</td>
</tr>
<tr>
<td>$K$</td>
<td>12</td>
<td>Number of options per period</td>
</tr>
<tr>
<td>$d$</td>
<td>6</td>
<td>Number of contacts for each agent</td>
</tr>
<tr>
<td>$\lambda_a$</td>
<td>1</td>
<td>Strength of innovation in choosing among contacts</td>
</tr>
<tr>
<td>$\lambda_b$</td>
<td>0.02</td>
<td>Strength of innovation in choosing between imitation and independence</td>
</tr>
</tbody>
</table>

Figure 6 depicts the resulting social hierarchy as it existed in the final period of the simulation. Across the top of the figure are the $K$ choices, labeled using lower-case letters. Individual agents, labeled using numbers, appear in rows below the choices based on the round of adoption. An arrow from an individual directly to one of the choices represents a choice by an individual who has acted independently for that period. An arrow from one agent to another represents a path of imitation with the arrow originating from the imitator.

In this example, agent #21 emerged from the population as the leader and a hierarchy forms based on agent #21’s leadership. In round 1 of each period subsequent to formation, agent #21, occupying tier 0 of the hierarchy, randomly selects one of the 12 options available for the period. In round 2, those in tier 1 of the hierarchy imitate agent #21. By round 5, agent #21’s choice for the period has disseminated through the entire population. In general, the rate of emergence depends on the values of $\lambda_a$ and $\lambda_b$ with increased values producing a shorter time for emergence.

An individual’s emergence is path dependent. In the early periods of the simulation, success by the individual is the result of random events that, in the absence of any adjustment, are transitory. For a leader to emerge, others in the population must respond to the lucky individual’s success by, for example, increasing the weight, $w'_{ia}$, of the successful contact and decreasing their own $\theta_{ia}$ in response to the higher payoff offered by imitation. The process of observation and adjustment allows the successful agent to become empowered by her followers, eventually generating a hierarchy. Success breeds further success.

Once a leader emerges, the employed RD process succeeds in generating the minimal hierarchical structure, $H_i$, based on the emergent leader $i$.

6. Conclusion

The steady state equilibrium of a social network is derived. The setting is one that rewards
individuals who act early in adopting a choice that eventually becomes popular. This incentive supports the existence of a structured hierarchy, comprised of a single leader and a network of followers. Each individual in the population imitates the choice of the leader, either through observing the leader directly or by imitating the leader indirectly through a chain of imitators.

Given a leader, a single minimal hierarchy uniquely optimizes the expected payoff for each individual in the population. While the realized position of each agent in the hierarchy depends on who in the population is the leader, the general leader/follower structure results regardless of who occupies the leadership position. Each non-leader in the population would prefer that someone else in the population serve as the leader (namely herself); however, given a leader, each follower in the population prefers her minimal position in the hierarchy over any other, given the minimal behavior of others. The agents also prefer the hierarchy to autarky.

The conditions that support the hierarchy are independent of the individual who emerges as a leader and the architecture of the minimal hierarchy. The environment favors a leader/follower hierarchy without dictating who should be leader.

A leader is empowered by her followers; a hierarchy is built upon the willing participation of those in the lowest tier, without whose participation a leader cannot ensure her choice will be subsequently popular. The equilibrium solution reveals that the hierarchy is supported by the utility payoff of following a popular trend, not only leading it, and by the existence of choice. The former rewards those occupying the lowest tier of the hierarchy, while the latter discourages independent choice in the hope of being an early adopter by selecting the same as the leader. An environment with insufficient reward to following a popular choice or insufficient number of options leads those at the bottom of the hierarchy to abandon their position in favor of acting independently.

While the equilibrium analysis presumes the existence of a leader around which a hierarchy forms, simulation of the dynamic model demonstrates that robust emergence of a leader is achieved with a simple and intuitive process of self-serving adjustment to observed payoffs.

The existence of only a single hierarchy in equilibrium is the result of assumptions that could be relaxed in order to generate multiple coexisting hierarchies in equilibrium. Already discussed is concavity in the conformity payoff. Reintroducing innate preferences would be another mechanism for generating multiple hierarchies. An individual will not join the hierarchy if she has a strong preference different from that of the leader.
There are a number of other issues to explore with the incorporation of innate preferences. Once the consumers have preference over the set of options they face, then there is a need on the part of the consumer to balance the two social aspects to consumption with their own innate preference. Once consumers are defined in a preference space, there is considerable opportunity to examine how these populations organize themselves and how leaders emerge to serve a population defined by their preferences. Noise and drift in innate preferences can also be introduced. Another modification that could act as a catalyst for change is to introduce cost for information gathering. Stability of a leader’s position, the sustainable size of a hierarchy, and how broad a population a leader can influence are all rich areas for investigation.

Bibliography


Appendices

Appendix A

Table 3 contains the information regarding the social network and payoffs of an example for which hierarchy $B$ falls under Case IV. Each of the four agents has the opportunity to act independently, a strategy indicated by “I”, or to imitate one of two contacts. For agents 1, 2, and 4, the contacts all consist of other agents within hierarchy $B$. Agent 3 can imitate 1 or join hierarchy $A$, also consisting of four members, $N_A' = 4$, the structure of which is not important. Assume that 3’s link to $A$ places her at the bottom of hierarchy $A$.

At the top of Table 3 are the numbered states. Under “strategy” are the strategies employed to achieve each state. The two lower sections include the state-dependent payoffs to each agent depending on whether agent 3 imitates agent 1 or joins hierarchy $A$, as indicated. The table contains all of the information of the extensive form tree, which is large but straightforward to construct. Each state requires the construction of the resulting hierarchy to determine individual payoff. Those payoff areas labeled “loop” are strategies that produce self-referencing imitation loops with no independent actor within the loop. Since this is presumed to produce a zero payoff, it is never an optimal strategy.

For convenience and without loss in generality, the payoffs entered in each cell of Table 3 include only the components of the payoff associated with participating in the hierarchy as it exists in the given state, $J(N_i') + T(N_i')$. Excluded is the expected payoff associated with the possible overlap in choice by independent actors. The true payoff would depend on $K$ through this possibility of overlap. Fortunately, under Condition B, the magnitude of the expectation of overlap component is insufficient to alter any of the relative values. Since Condition B is already presumed to hold in this setting, the expected overlap component of the payoff can be ignored without altering the game. Conveniently, this implies that equilibrium subgame perfect strategy is not affected by $K$.

The analysis based on the extensive form game in which the decision order proceeds 1, 2, 4, 3. It is never optimal for agent 3 to act independently. Eliminating this strategy from the strategy tree leaves 54 possible states. These are spread out over two sets of columns of 27

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20 The cell entries are correct for $K \rightarrow \infty$, in which case the probability of overlap converges to zero.
payoffs each; the first set in the upper portion of Table 3 based on agent 3 imitating agent 1 and the second set in the lower portion based on agent 3 joining $A$.

Consider only the top section of payoffs based on agent 3 imitating agent 1. Analysis of these payoffs confirms that the hierarchical structure for $B$ in Figure 3, as is produced at state 5, is a Nash equilibrium. Given the strategy choice of the other agents, each agent employs the optimal strategy. This includes agent 3 who chooses to imitate 1 to keep agent 4 as a follower rather than acting on the option to join at the bottom of hierarchy $A$ to form a larger hierarchy.

Table 3: Example of Case IV for considering a multiple hierarchy equilibrium

<table>
<thead>
<tr>
<th>Agent</th>
<th>Payoff &quot;3&quot; imitates 1</th>
<th>Payoff &quot;3&quot; joins A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1 1.1 2.2 2.2 3.3 3.3 1.1</td>
<td>Loop</td>
</tr>
<tr>
<td>2</td>
<td>0 1.1 0 0.2 1.3 1.3 0.1</td>
<td>Loop</td>
</tr>
<tr>
<td>3</td>
<td>0 0.3 0.2 0.3 0.3 1.1</td>
<td>Loop</td>
</tr>
<tr>
<td>4</td>
<td>0 0.1 0.1 0.1 0.5 0.2</td>
<td>Loop</td>
</tr>
</tbody>
</table>

Examine the subgame perfect strategies of 1, 2, and 4, assuming agent 3 remains in $B$ (the top payoff section only). To assist visualizing the equilibrium behavior, the subgame optimal choices of each agent are shaded (if viewed in color, they are blue for agent 4, orange for agent 2, and yellow for agent 1). State 5 (along with the equivalent state 6) is a subgame perfect equilibrium when agent 3 is constrained to imitate agent 1.

State 5 is not a subgame perfect equilibrium when agent 3 can choose $A$. Given the strategy of agent 1 choosing “I” and agent 2 choosing “1” that lead to states 4, 5, and 6, agent 4 optimally chooses to imitate agent 3 and agent 3 optimally chooses to join $A$. To both 3 and 4’s benefit, instead of receiving the state 5 payoff of $\{0.3, 1.3\}$ from the upper payoff table, they receive the state 6 payoffs $\{0.5, 1.5\}$ from the lower payoff table.

Analysis of the extensive form game reveals that the subgame perfect equilibrium is for agent 3 to join $A$, for agents 1 and 4 to imitate agent 3, and for agent 2 to imitate either agent 1 or
4. These are states 15 and 18 in Table 3. The sub-hierarchy under agent 3 within $H_A$ is depicted in Figure A.1.

**Figure A.1** Sub-hierarchy of $H_A$ as linked through agent 3

![Sub-hierarchy of $H_A$](image)

**Appendix B**

The social network is as listed in Table 1 of the main paper. If Condition B is met, as followers, $o_1$ and $o_2$ always choose to follow $j$. The extensive form game only includes the decisions of $i, j,$ and $n$ in that order of play. Each agent has the option to act independently, labeled “$I$”, or to imitate the first or second contact. The payoff to each agent in each state is listed in the “payoff” section of Table 4.

State 5 in the decision tree generates the minimal hierarchy under $i, H_i$. The hierarchy $H_j$ is produced by both 12 and 20. The subgame perfect state dependent strategy of each agent is shaded (if viewed in color, they are colored blue for agent $n$, orange for agent $j$, and yellow for agent $i$).

As can be seen, both states 12 and 20 generating hierarchy $H_j$ are subgame perfect solutions to this extensive form game while 5 is excluded.

**Table 4:** $H_i$ (state 5) as a Nash equilibrium but not subgame perfect (states 12 and 20)

| Agent | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $i$   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| $j$   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| $n$   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |

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Appendix C
Proof of Proposition 3: For contact $d1$ occupying tier $\delta_1$ and contact $d2$ occupies tier $\delta_2$ with $\delta_1 < \delta_2$, with $T'(x) > 0$, for $N^T_\theta | d1_j > N^T_\theta | d2_j$, the contact offering the highest tier is the optimal choice. For $N^T_\theta | d1_j = N^T_\theta | d2_j$, the lower tier has to be eliminated as an option agent $j$ will choose. [I am still working this. A case can be made based on the lower tier’s weaker position against perturbations to equilibrium play by others in the population. If the alternate path is the same as the minimal path plus one additional agent, then the alternate exposes agent $j$ to greater threat from perturbations from equilibrium play that includes non-participation. At present, eliminating from the equilibrium an alternate path that does not include the full minimal path as a component is more problematic. If no other option to eliminate the alternates become available, I can do so by decide to including a small payment to agent $j$ for those that act contemporaneous to her, though it is not attractive.]
Figure 5: Time-series of population characteristics. Replicator Dynamics, $K = 12$, $\lambda_d = 1$, $\lambda_a = 0.02$. $N(\text{ind}) =$ number of agents acting independently; $\text{MaxFans} =$ maximum number of fan linking to any one agent; $N(r=tr) =$ number of agents who do not decide until the final round (an indication that the agent decided to imitate in the round, but did not have a direct or indirect link to someone who acted independently through the social network); Optimality = number of potential fans of the leader – maximum number of potential fans across the population; Efficiency is the cumulative number of tiers each agent is below her minimal position. The leader emerges as sole leader, the only agent acting independently. Soon after, the efficiency of the hierarchy increases to zero. The leader is not a natural leader. Once established, there are no disruptions to the hierarchy.
**Figure 6:** Structure of the social hierarchy in the final period (t = 500). Replicator Dynamics, $K = 12$, $\lambda_A = 1$, $\lambda_B = 0.02$. Minimal hierarchy based on Agent #21, who emerged from the population as the leader.