Auctions with a Buy Price*

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Abstract

Internet auctions on eBay and Yahoo allow sellers to list their auctions with a *Buy-Now* option. In such auctions the seller sets a *buy price* at which a bidder may purchase the item immediately and end the auction. In the eBay version of a buy-now auction, the buy-now option disappears as soon as a bid is placed, while in the Yahoo version of the auction the buy-now option remains in effect throughout the auction. We show that when bidders are risk neutral then both auction formats are revenue equivalent to the standard English ascending bid auction, so long as the buy-now price is not too low. When bidders are risk averse, these auction formats are advantageous for the seller since for a wide range of buy prices each auction format raises more revenue than the ascending bid auction. We show that while the Yahoo format raises more revenue than the eBay format (when the buy-now price is the same in both auctions), the auctions are payoff equivalent from the bidders' perspective.

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1 Introduction

The expansion of commerce conducted over the Internet has sparked a surge of interest in auctions and new auction forms. Many online auction sites appeared, and quite a few subsequently disappeared. These online auctions use a variety of formats and rules. In many cases online auctions adapted procedures that had been used for running auctions long before the Internet came into being. In a few cases online auctions introduced features which appear to be new and unique to the online environment. Lucking-Reiley (2000) describes the wide variety of online auction formats that were being used as of 1999.

An example of a new twist in online auction formats appears in Yahoo and eBay auctions. In 1999 Yahoo introduced the *Buy-Now* feature into its ascending bid auctions. The Buy-Now feature allows the seller to set a price, termed a buy price, at which any bidder may purchase the item at any time during the auction.[1] Since the buy price remains in effect throughout the auction, this feature allows the seller to post a maximum price for the item. In 2000 eBay introduced its own version of a fixed price feature into its online auctions via the *Buy It Now* option. In contrast to the Yahoo format, eBay permits bidders to select the buy price only at the opening of the auction, before any bids are submitted, or in the case of an auction with a (secret) reserve, before bids reach the reserve price.[2] We use the expression "buy-now auction" as the generic term for an ascending bid auction with a buy price.

The buy-now auction format has become quite popular at both eBay and Yahoo. Table I lists the total number of auctions and the number of buy-now auctions on eBay and Yahoo in selected categories on a recent day (the categories are similar, but not identical across the two auction sites). Overall, about 40% of these eBay auctions and 66% of these Yahoo auctions utilize the buy-now feature. Hof (2001) also cites a

[1] For more on this format see, http://auctions.yahoo.com/phtml/auc/us/promo/buynow.html

[2] See, http://pages.ebay.com/services/buyandsell/buyitnow.html

40% figure for the fraction of eBay auctions that use the buy-now feature.

	Yahoo Auctions		eBay Auctions	
Category	**# buy-now**	**total #**	**# buy-now**	**total #**
Automobiles	157	186	662	2,378
Clothing	148	223	2,678	10,078
DVD Players	86	115	981	2,251
VCR's	91	136	384	1,078
Digital Cameras	337	583	7,179	13,284
TV's	23	39	596	2,073

Table I: Buy-Now and Total Auctions for Yahoo and eBay
(for auctions on March 27, 2002)

At first glance the prevalence of buy-now auctions is puzzling. After all, an ascending bid auction is intended to elicit high bids from potential buyers. Putting a cap on these bids (as in a Yahoo buy-now auction) or offering a fixed price at the auction open (as in an eBay buy-now auction) would seem to limit the seller's expected revenue. But there are at least two reasons why a buy-now auction might yield greater revenue for the seller. First, offering a buy price may reduce the riskiness of the auction for bidders. Consider a bidder whose value exceeds the buy price. If the bidder does not accept the buy price then he may win the item at a price less than the buy price, but he runs the risk of losing the auction (this would happen if another bidder accepts the buy price, or if he is outbid in the ascending auction). A risk-averse bidder would be willing to pay a buy price which includes a risk premium, rather than face uncertainty regarding whether he wins and how much he pays. Second, with the buy-now format the seller might be able to exploit impatient bidders who are willing to pay a premium included in the buy price in order to end the auction early.

In this paper we analyze the eBay and Yahoo buy-now auctions, focusing on the effects of bidder risk aversion rather than on bidder impatience. We utilize a symmetric independent private values framework with a continuous distribution of values for bidders. In both auction formats, in the first stage the bidders simultaneously choose whether to accept the buy price or wait. If a bidder accepts the buy price, then the auction ends and he pays the buy price. In the eBay auction, if the bidders wait then the buy-now option disappears and, in the ascending bid auction that follows, the bidder with the highest value wins the item and pays the second highest value. In the Yahoo auction, if the bidders wait then there is an ascending clock auction; as the clock progresses, at any point a bidder may either remain in the auction, drop out of the auction, or accept the buy price. When all bidders except one drop out, then the remaining bidder wins the auction and pays the current bid. If any bidder accepts the buy price then he wins the item and pays the buy price. In the Yahoo auction the buy price is a price cap since the buy-now option remains in effect throughout the auction, while in the eBay auction the final price may exceed the buy price since the buy-now option disappears after bidding begins.

We characterize equilibrium in both auctions when bidders are risk-neutral and when bidders are risk averse. Equilibrium strategies in the eBay auction are characterized by a "cutoff" value; a bidder accepts the buy price at the auction open if their value exceeds the cutoff and waits otherwise. In the Yahoo auction equilibrium is characterized by threshold strategies which specify the bid price (or, clock time) at which a bidder will accept the buy price, as a function of his value.

Section 3 considers the case where bidders are risk neutral. We show that the eBay and Yahoo buy-now auctions are revenue equivalent. Moreover, the set of bidder types which accepts the buy price immediately is the same in the both auctions. If the buy price is above the mean of the value distribution then the buy price is never accepted in the eBay auction, but it is accepted with positive probability in the ascending clock phase of the Yahoo auction. We also compare the revenue of the eBay and Yahoo buy-now auctions to the revenue of the standard ascending bid auction. If the buy price is above the mean of the value distribution, then all three auctions are revenue equivalent, but otherwise the buy-now auctions yield less revenue.

Section 4 deals with bidders who have constant absolute risk aversion. In this case there is a critical buy price (which exceeds the mean of the value distribution). If the buy price is above the critical price then it is not accepted immediately in either the eBay or the Yahoo auction. If the buy price is below the critical price, then it is accepted with positive probability at the auction open in both the eBay and Yahoo auction. (In fact, the set of bidder types which accepts the buy price immediately is the same in both auctions.) We show that both auction formats yield more revenue to the seller than the standard ascending bid auction, for a wide range of buy prices. The seller is not indifferent between the two auction formats: the Yahoo auction raises more revenue than the eBay auction whenever the buy price is the same in both auctions. On the other hand, the two auction formats are payoff equivalent for the bidders.

We are aware of one prior analysis of auctions with a buy price. Budish and Takeyama (2001) analyze the Yahoo buy-now auction in a model with two bidders and two possible valuations for each bidder – high or low. They demonstrate that when bidders are risk averse, then there is a buy price for which bidders with the high-value accept immediately, bidders with the low-value wait, and which yields more revenue to the seller than the ascending bid auction.[3] Our analysis, with a continuum of bidder values, yields insights into the relationship between a bidder's value and the threshold at which he accepts the buy price in the ascending bid phase of the auction. In contrast to Budish and Takeyama, we find that the buy-now auction may raise seller revenue even if it is not accepted immediately by any bidder type. Our analysis also provides a comparison of the eBay and Yahoo versions of the buy-now mechanisms.

Several recent papers have analyzed online auctions. Roth and Ockenfels (2002a) compare eBay and Amazon auctions in order to evaluate the effect that differences in the rules for ending an auction have on bidding behavior and auction outcomes. Peters and Severinov (2002) analyze the equilibrium strategies of sellers and bidders when many auctions are conducted simultaneously, as is often the case for online auctions websites. Bajari and Hortascu (2002) document empirical regularities in a sample of eBay coin auctions and estimate a structural model of bidding on eBay.

[3] In Budish and Takeyama the highest buy price for which it is an equilibrium for high-value bidders to accept immediately and low-value bidders wait is denoted by B^*. They claim that this is the seller's optimal buy price, without considering the revenue consequences of a buy price above B^*.

2 The Model

There are two bidders for a single item whose values are independently and identically distributed according to a uniform distribution with support $[\underline{v}, \bar{v}]$ and *c.d.f.* F. Denote by v_i the value of bidder i. Let B denote the buy price set by the seller. We assume that $\underline{v} < B < \bar{v}$, since a seller would never wish to set $B \leq \underline{v}$, whereas if $B \geq \bar{v}$ then no bidder will ever take it. We consider auctions in which there is no reserve. If a bidder whose value is v wins the item and pays price p then his payoff is $u(v-p)$; he obtains a payoff of zero otherwise. Section 3 deals with the case where both bidders are risk neutral, i.e., $u(v-p) = v-p$. Section 4 deals with the case where both bidders have constant absolute risk aversion (CARA), each with $u(x) = 1 - e^{-\alpha x}$.

eBay

At the open of the auction the bidders simultaneously decide whether to "buy" or "wait." If a bidder chooses to buy, then he wins and he pays the seller B. (If both bidders buy then the winner is randomly assigned.) The bidding process that follows if both the bidders wait is not explicitly modeled. Instead, we suppose that if both the bidders wait, then the buy-now option disappears, and the final outcome is that the bidder with the highest value wins the item and pays the second highest value.[4] This is consistent with there being, for example, either an ascending clock auction or a second-price sealed-bid auction if both bidders wait.

In the eBay auction a bidder's strategy tells him, for each possible value, whether to buy or wait. We focus on equilibria in "cutoff strategies." A **cutoff strategy** for a bidder is characterized by a value $c \in [B, \bar{v}]$ such that he chooses buy if his value exceeds c and chooses wait if his value is below c. Suppose that bidder i's value is v_i and his rival's cutoff is c. If bidder i chooses buy, then he wins the auction if his rival waits (which occurs with probability $F(c)$) and he wins the auction with probability

[4]Here we deal with auctions in which there is no reserve, in which case the buy-now option disappears as soon as a bid is placed. In eBay auctions with a reserve, the buy-now option disappears only once there is a bid placed about the reserve price.

$\frac{1}{2}$ if his rival also chooses buy (this occurs with probability $\frac{1}{2}[1 - F(c)]$). Bidder i's expected payoff if he chooses buy is

$$U^b(v_i, c) = \left[\frac{1}{2}(1 - F(c)) + F(c)\right] u(v_i - B).$$

If bidder i waits, then he wins the auction only if his rival also waits (i.e., $v_j < c$) and his value is higher than his rival's (i.e., $v_j < v_i$). His expected payoff is

$$U^w(v_i, c) = \int_{\underline{v}}^{\min\{v_i, c\}} u(v_i - v_j) dF(v_j).$$

A cutoff c^* is a **symmetric equilibrium** if $U^w(v_i, c^*) > U^b(v_i, c^*)$ for all $v_i \in [\underline{v}, c^*)$ and $U^w(v_i, c^*) < U^b(v_i, c^*)$ for all $v_i \in (c^*, \bar{v}]$. That is, given that bidder j uses the cutoff c^* then it is optimal for bidder i to wait if $v_i < c^*$ and it is optimal for bidder i to buy if $v_i > c^*$.

Yahoo buy-now Auctions

We model the Yahoo auction as an ascending clock auction, in which the bid rises continuously from $\underline{v}$ to B. As the clock progresses, at any point a bidder may either drop out of the auction or may accept the buy price. If he drops out when the bid is b, then his rival wins the auction and pays b. If a bidder accepts the buy price, then he wins the auction, he pays B, and the auction ends.

Clearly a bidder whose value is less than B never accepts the buy price since by doing so he obtains he obtains a negative payoff, whereas he would obtain a payoff of zero by dropping out. We assume that such bidders simply drop out when the bid reaches their value. Similarly, a bidder whose value is above B never drops out since whatever the current bid is, he obtains a positive payoff accepting the buy price but would obtain zero by dropping out. Thus we focus on how bidders whose values are above B choose the bid at which to accept the buy price. A strategy for a bidder is a function which gives for each value v in $[B, \bar{v}]$ a threshold bid price $t(v)$ (in $[\underline{v}, B]$) at which the bidder accepts the buy price. A function $t : [B, \bar{v}] \to [\underline{v}, B]$ is a **threshold strategy** if either (i) t is continuous and strictly decreasing on $[B, \bar{v}]$, or (ii) there is a $z \in (B, \bar{v})$ such that t is continuous and strictly decreasing on $[B, z]$, and t jumps down to $t(v) = \underline{v}$ for $v \in (z, \bar{v}]$. If $t(v) = \underline{v}$ the bidder with value v accepts the buy price at the auction open.

To see how the bidders' payoffs are determined given a profile of threshold strategies, it is useful to consider Figure 1 which shows bidder 2's value on the horizontal axis. Suppose bidder 2 follows the threshold strategy $t(v)$. Let $[\underline{t}, \bar{t}]$ denote the range of threshold values for which $t(v)$ is strictly decreasing. (For $t(v)$ as in the figure, $\bar{t} = B$ and $\underline{t} = t^{-1}(\bar{v})$.) Suppose bidder 1 chooses the threshold $\tilde{t}$. If bidder 2's value v_2 is below $\tilde{t}$, then bidder 2 drops out when the bid reaches v_2, bidder 1 wins, and bidder 1 pays v_2. If bidder 2's value is above $\tilde{t}$ but below $t^{-1}(\tilde{t})$, then bidder 1 accepts the buy price when the bid reaches $\tilde{t}$, he wins the auction, and he pays B. Finally, if bidder 2's value is above $t^{-1}(\tilde{t})$, then bidder 2 accepts the buy price when the bid reaches $t(v_2)$, he wins the auction, and he pays B.

Figure 1 goes here.

Hence, if a bidder's value is v, he chooses the threshold $\tilde{t}$, and the other bidder follows the threshold strategy t (one without a jump down) then the bidder's expected utility is

$$U(\tilde{t}, v; t) = \begin{cases} \int_{\underline{v}}^{\tilde{t}} u(v-x)dF(x) + [F(t^{-1}(\tilde{t})) - F(\tilde{t})]u(v-B) & \text{if } \tilde{t} \in [\underline{t}, \bar{t}] \\ \\ \int_{\underline{v}}^{\tilde{t}} u(v-x)dF(x) + [1 - F(\tilde{t})]u(v-B) & \text{if } \tilde{t} < \underline{t}. \end{cases}$$

Note that if $\tilde{t} < \underline{t}$ then the bidder wins for sure, paying the other bidder's value when the value is less than $\tilde{t}$ and paying B otherwise.

We say that t is a **(symmetric) equilibrium in threshold strategies** if for each $v \in [B, \bar{v}]$ we have

$$U(t(v), v; t) \geq U(\tilde{t}, v; t) \quad \forall \tilde{t} \in [\underline{v}, B].$$

In other words, for each value v a bidder's optimal threshold is $t(v)$ when the other bidder follows the threshold strategy t.

While these models capture salient features of buy-now auctions as they are implemented on eBay and Yahoo, there are some differences. eBay auctions end at a predetermined time specified by the seller (i.e., they have a "hard" close). Roth and Ockenfels (2002b) show that a hard close, combined with uncertain processing of proxy bids placed in the last minutes of an auction, can lead to a final price less than the second highest value.[5] Yahoo auctions are not conducted as ascending clock auctions, but rather allow bidders to enter either a fixed bid or to make proxy bids. Last, here we have supposed that there is a fixed commonly known number of bidders, a condition that is unlikely to prevail in actual Internet auctions.

3 Risk Neutral Bidders

In this section we study the eBay and Yahoo buy-now auctions when the bidders are risk neutral. Let $\bar{B} = \frac{1}{2}(\underline{v} + \bar{v})$ denote the mean of the uniform distribution on $[\underline{v}, \bar{v}]$.

eBay buy-now auctions

Proposition 1 indicates that for the eBay buy-now auction the equilibrium and efficiency results can be divided into two cases, depending on whether the buy price is high or low.

Proposition 1: *Suppose the bidders are risk neutral.*
(i) If $B > \bar{B}$ then the buy price is never accepted by a bidder (i.e., the unique symmetric equilibrium cutoff value is $c^* = \bar{v}$).
(ii) If $B < \bar{B}$ then there is a unique symmetric equilibrium cutoff value of

$$c^* = \frac{B(\bar{v} - 2\underline{v}) + \underline{v}^2}{\bar{v} - B}. \tag{1}$$

The cutoff value is increasing in the buy price. Equilibrium is inefficient since the item is awarded to the low-value bidder with positive probability.
Proof: Appendix.

Proposition 1(i) is intuitive: A bidder with the highest possible value $\bar{v}$ who rejects the buy price pays $\bar{B}$ (in expectation) in the eBay auction if his rival always rejects

[5] See http://pages.ebay.com/help/buy/proxy-bidding.html for a description of proxy bidding.

the buy price. Hence, if $B > \bar{B}$ then such a bidder obtains a higher payoff by rejecting the buy price than accepting it. If a bidder with the highest value optimally rejects the buy price, then bidders with lower values will also optimally reject. Hence in equilibrium both bidders (and all types) reject the buy price when $B > \bar{B}$.

The buy-now auction is inefficient when the buy price is set low enough so that some bidder types accept it. The inefficiency is similar to the inefficiency that results when a single item is offered for sale at a fixed price to multiple buyers. If there is no mechanism to put the high-value buyer at the head of the queue of buyers, then there is a positive probability that the high-value buyer will not receive the item. The buy-now auction is efficient only when the buy price is set so that no bidder types accept it.

Yahoo buy-now auctions

Our next proposition characterizes the symmetric equilibrium threshold function in Yahoo buy-now auctions.

Proposition 2: *If bidders are risk neutral then there is a unique symmetric equilibrium $t(v)$ in threshold strategies that are differentiable (except possibly at one point where the threshold strategy jumps down).*
(i) If $B \geq \bar{B}$ then $t(v) = 2B - v$. The equilibrium is efficient.
(ii) If $B < \bar{B}$, then

$$t(v) = \begin{cases} 2B - v & \textit{if } v \leq \frac{B(\bar{v}-2\underline{v})+\underline{v}^2}{\bar{v}-B} \\ \underline{v} & \textit{otherwise.} \end{cases}$$

The equilibrium is inefficient.
Proof: Appendix.

In contrast to the eBay auction, in the Yahoo auction a buy price B is accepted by some bidder types even if $B > \bar{B}$. The Yahoo auction is nonetheless efficient in this case since the equilibrium threshold function is strictly decreasing, and hence the first bidder to accept the buy price has the highest value. If $B < \bar{B}$ then there is a positive probability that both bidders buy immediately at the auction open and hence the auction is inefficient.

By Proposition 2(ii), if $B < \bar{B}$ then the value at which the threshold function jumps down is c^* (see (1)), which is the same as the symmetric equilibrium cutoff value in the eBay auction. Hence, the ex-ante probability that a bidder accepts the buy price at the auction open is the same for the Yahoo and eBay auctions, given a common $B < \bar{B}$.

Comparing Auctions

We now show that the seller's expected revenue is the same in the eBay and the Yahoo auction when the buy price is the same in both auctions. Figure 2(a) below shows the equilibrium threshold function when $B > \bar{B}$. Let bidder 1's value v_1 be fixed and assume that $v_2 \leq v_1$. If $v_1 \leq B$, then seller revenue is v_2 in both the Yahoo and eBay auctions. If $v_1 > B$ then revenue is still v_2 in both auctions if $v_2 \leq 2B - v_1$. If, however, $v_2 > 2B - v_1$ then the seller gets B for sure in the Yahoo auction; in the eBay auction seller revenue is v_2, but the expected value of v_2 conditional on $v_2 \in [2B - v_1, v_1]$ equals B.[6] We have shown that for each fixed v_1, given $v_2 \leq v_1$ the seller's expected revenue is the same in both auctions. Since this is true for each v_1, the seller's unconditional expected revenue must be the same for both auctions.

Figure 2 goes here.

Figure 2(b) shows the equilibrium threshold function in the Yahoo auction when the threshold function jumps down. Again, fix v_1 and assume that $v_2 \leq v_1$. The key observation when comparing revenues in the two auctions is that the value at which a bidder's threshold function jumps down in the Yahoo auction is the same as the bidder's cutoff value in the eBay auction. Hence if $v_1 > c^*$ then seller revenue is B in both auctions. If $v_1 \leq c^*$, then the same argument as in the $B > \bar{B}$ case establishes that expected seller revenue is the same in both auctions. This establishes the following Corollary.

[6] Given $v_2 \in [2B - v_1, v_1]$, then v_2 is conditionally distributed $U[2B - v_1, v_1]$. Since the mean of the $U[a, b]$ density is $\frac{a+b}{2}$, the conditional expectation of v_2 is B.

Corollary 1: *Assume that the bidders are risk neutral. Equilibrium expected seller revenue is the same in the Yahoo buy-now and eBay buy-now auctions when the buy price is the same in both auctions.*

The variance of seller revenue, however, is lower in the Yahoo auction. Hence a risk averse seller would prefer the Yahoo to the eBay auction, when the buy price is the same in both auctions.

It's not difficult to see that the two auctions are payoff equivalent from the perspective of risk-neutral bidders. If bidder i's value is $v_i < c^*$ then his expected payoff in the eBay and Yahoo auctions are equal, i.e.,

$$\int_{\underline{v}}^{v_i} (v_i - v_j)\, dF(v_j) = \int_{\underline{v}}^{2B-v_i} (v_i - v_j)\, dF(v_j) + \int_{2B-v_i}^{v_i} (v_i - B)\, dF(v_j).$$

If $v_i > c^*$ then the bidder's expected payoff is $\frac{1}{2}(1 + F(c^*))(v_i - B)$ in both auctions. Hence the auctions are *interim* (and *ex-ante*) payoff equivalent.

While the eBay and Yahoo auctions are payoff equivalent for a risk-neutral seller and for risk-neutral bidders, the *ex-post* outcomes may be quite different. If $B \geq \bar{B}$ then no bidder type buys in the eBay auction, while some bidder types buy in the Yahoo auction. If $B < \bar{B}$ then the set of value combinations for which the buy price is accepted immediately is the same in both auctions. In the Yahoo auction, however, the buy price may also be accepted in the ascending bid phase of the auction. In the eBay auction the price exceeds B with positive probability, while in the Yahoo auction it never exceeds B.

We conclude this section with a comparison of seller revenue in the eBay, Yahoo, and English ascending bid auction.

Corollary 2: *Assume that the bidders are risk neutral. If* $B > \bar{B}$ *then the seller's expected revenue is the same in the eBay, Yahoo, and English ascending bid auction. If* $B < \bar{B}$ *(and so the buy price is accepted with positive probability in the eBay and Yahoo auction) then expected seller revenue in the eBay and Yahoo buy-now auction is less than expected seller revenue in the English auction.*

Proof: Appendix.

4 Risk Averse Bidders

In this section we provide several results on the effects of risk aversion on bidder behavior and on seller revenue. We focus on the case of constant absolute risk aversion (CARA), since preferences of this type yield tractable equilibrium conditions. We suppose that both bidders have CARA utility of the form $u(x) = 1 - e^{-\alpha x}$, where $\alpha > 0$ is the index of bidder risk aversion. For $\alpha > 0$ define

$$B_\alpha = \frac{1}{\alpha} \ln \left(\frac{e^{\alpha \bar{v}} - e^{\alpha \underline{v}}}{\alpha(\bar{v} - \underline{v})} \right). \tag{2}$$

It is straightforward to verify that $B_\alpha > \bar{B}$.[7] For a bidder whose value is $\bar{v}$, B_α is the certainty equivalent of the price he would pay in a second price auction.[8]

The following Lemma is useful in characterizing equilibrium. It states that for each buy price B there is a unique cutoff value which makes a bidder, whose value equals the cutoff, indifferent between buying or waiting in the eBay auction.

Lemma 1: *(i) For each buy price B, where $\underline{v} \leq B \leq B_\alpha$, there is a unique c_α* in *$[B, \bar{v}]$ satisfying*

$$U^b(c_\alpha, c_\alpha) = \frac{1}{2}(1 + F(c_\alpha))u(c_\alpha - B) = \int_{\underline{v}}^{c_\alpha} u(c_\alpha - v_j)dF(v_j) = U^w(c_\alpha, c_\alpha). \tag{3}$$

Moreover, c_α is strictly increasing in B, $c_\alpha = \underline{v}$ if $B = \underline{v}$, and $c_\alpha = \bar{v}$ if $B = B_\alpha$. If $B > B_\alpha$ then there is no c_α in $[B, \bar{v}]$ satisfying (3).
(ii) If $B < B_\alpha$ then c_α is strictly less than c^* (see (1)).
Proof: Appendix.

[7]Since $e^x > 1 + xe^{x/2}$ for $x > 0$, choosing $x = \alpha(\bar{v} - \underline{v})$ and rearranging yields

$$\frac{e^{\alpha \bar{v}} - e^{\alpha \underline{v}}}{\alpha(\bar{v} - \underline{v})} > e^{\alpha(\bar{v} + \underline{v})/2}.$$

Taking logs of both sides and rearranging gives the result.

[8]In other words,

$$u(\bar{v} - B_\alpha) = \int_{\underline{v}}^{\bar{v}} u(\bar{v} - v_j)dF(v_j).$$

Proposition 3 characterizes the symmetric equilibrium cutoff value for the eBay auction when bidders are risk averse. It shows that B_α, defined in (2), is a critical price; if the buy price exceeds this price then it is never accepted in equilibrium.

Proposition 3: *Assume bidders are CARA risk averse with index of risk aversion* $\alpha > 0$. *For each* B *there is a unique symmetric equilibrium in cutoff strategies.*
(i) If $B < B_\alpha$ *then the symmetric equilibrium cutoff value is* c_α*, the solution to (3);*
(ii) If $B \geq B_\alpha$ *then the symmetric equilibrium cutoff value is* $\bar{v}$*, i.e., the buy price is never accepted by a bidder.*
Proof: Appendix.

Risk aversion changes the bidders' willingness to accept the buy price. Acceptance of the buy price does not completely eliminate uncertainty for a bidder, because there is a chance that the other bidder will also accept; the object is randomly awarded in this case. But acceptance of the buy price does reduce the chance that a bidder will "lose" (i.e., not be awarded the object) and have a zero surplus. A risk averse bidder is willing to trade off this reduced chance of losing (and receiving the lowest possible surplus, zero) for a lower expected surplus. Note that the maximum buy price accepted when bidders are risk averse (i.e., B_α) is greater than the maximum buy price accepted when bidders are risk neutral (i.e., $\bar{B}$).

The increased willingness of risk averse bidders to accept a buy price translates into more revenue for the seller in an eBay auction. Suppose the buy price is between $\bar{B}$ and B_α (and so $c_\alpha < c^* = \bar{v}$). Let bidder 1's value be fixed and suppose $v_2 \leq v_1$. If $v_1 < c_\alpha$ then seller revenue is v_2 and if $v_1 > c^*$ then seller revenue is B, whether bidders are risk averse or risk neutral. If $c_\alpha < v_1 < c^*$ then seller revenue is v_2 if bidders are risk neutral and is B if bidders are risk averse. Thus seller revenue is higher if bidders are risk averse since

$$E[v_2 | v_2 \leq v_1] = \frac{\underline{v} + v_1}{2} \leq \frac{\underline{v} + \bar{v}}{2} = \bar{B} < B.$$

If $B < \bar{B}$ then $c_\alpha < c^* < \bar{v}$. As before, if $v_1 < c_\alpha$ or $v_1 > c^*$ then revenue is the same whether bidders are risk averse or risk neutral. To deal with the case $c_\alpha < v_1 < c^*$ it is useful to note that $B < \bar{B}$ implies $c^* < 2B + \underline{v}$. Hence $v_1 < 2B + \underline{v}$, or $\underline{v} < 2B - v_1$. Thus seller revenue is higher if bidders are risk averse since

$$E[v_2 | \underline{v} \leq v_2 \leq v_1] < E[v_2 | 2B - v_1 \leq v_2 \leq v_1] = B.$$

Therefore, we have established the following.

Corollary 3: *If $B < B_\alpha$ then expected seller revenue in the eBay buy-now auction with CARA bidders exceeds expected seller revenue with risk neutral bidders. If $B > B_\alpha$ then the buy price is not accepted and seller revenue is the same whether bidders are risk averse or risk neutral.*

Yahoo buy-now auctions

Now we examine the effects of bidder risk aversion in the Yahoo buy-now auction. The following proposition gives the equilibrium threshold function when bidders are CARA risk averse. In the proposition $L^W(x)$ is the Lambert-W function.[9]

Proposition 4: *Assume bidders are CARA risk averse with index of risk aversion $\alpha > 0$. There is a unique symmetric equilibrium $t_\alpha(v)$ in threshold strategies that are differentiable (except possibly at one point where the threshold strategy jumps down).*
(i) If $B \geq B_\alpha$ then

$$t_\alpha(v) = B + \frac{1}{\alpha} \ln \left(-L^W \left[-e^{\alpha(v-B) - e^{\alpha(v-B)}} \right] \right) \tag{4}$$

for $v \in [B, \bar{v}]$.
(ii) If $B < B_\alpha$ then $t_\alpha(v)$ is as in (4) for $v \in [B, c_\alpha]$ and $t_\alpha(v) = \underline{v}$ for $v \in (c_\alpha, \bar{v}]$.
Proof: Appendix.

Note that the value at which the threshold function jumps down in the Yahoo auction (c_α) is also the equilibrium cutoff in the eBay auction. Consequently, the set of types which accept the buy price at the auction open is the same in the eBay and Yahoo auctions.

[9] For x satisfying $0 > x \geq -e^{-1}$ the equation $we^w = x$ has two solutions: $w = \text{LambertW}(x) \equiv L^W(x)$ and $w = \text{LambertV}(x)$. See http://www.mupad.com/doc/eng/stdlib/lambert.shtml.

In the proof of Proposition 4 we show that the equilibrium threshold function defined in (4) is strictly concave. Moreover, since $\lim_{v \downarrow B} t'_\alpha(v) = -1$ and since the threshold function when bidders are risk neutral is linear with slope -1, then the risk-averse threshold lies below the risk-neutral threshold except when both thresholds equal $\underline{v}$; in other words, risk-averse bidders accept the buy price at lower bid prices than risk-neutral bidders. This establishes Corollary 4(i) below.

Next we show that if $B > \bar{B}$, then seller revenue is higher in the Yahoo auction when bidders are risk averse than when they are risk neutral. Figure 3 illustrates the risk-averse bidding threshold and the risk-neutral bidding threshold when $B \in (\bar{B}, B_\alpha)$.

Figure 3 goes here.

If (v_1, v_2) lies above the risk-neutral bidding threshold $t(v_1)$ then seller revenue is B, whether bidders are risk averse or risk neutral. If (v_1, v_2) lies below the risk-averse threshold function $t_\alpha(v_1)$ then seller revenue is v_2, whether bidders are risk averse or risk neutral. However, if (v_1, v_2) lies between $t_\alpha(v_1)$ and $t(v_1)$ then seller revenue is B if bidders are risk averse but is only v_2 if bidders are risk neutral. Thus seller expected revenue is higher with risk-averse bidders than with risk neutral bidders when $B \in (\bar{B}, B_\alpha)$. The same argument applies for $B > B_\alpha$ or $B < \bar{B}$, which establishes Corollary 4(ii).

Corollary 4: *Assume bidders are CARA risk averse with index of risk aversion* $\alpha > 0$.

(i) If $B > \bar{B}$ *then* $t_\alpha(v) < t(v)$ *for all* $v \in (B, \overline{v}]$, *i.e., the symmetric equilibrium threshold function for CARA bidders lies below the symmetric equilibrium threshold for risk-neutral bidders;*

(ii) For any $B > \underline{v}$, *expected seller revenue in the Yahoo auction with CARA bidders exceeds expected seller revenue in the Yahoo auction with risk neutral bidders.*

Comparing Auctions

Our results so far imply that both the eBay and Yahoo buy-now auction format raise more revenue for the seller than an English ascending bid auction for suitably chosen buy prices. Write R^{eBay}, R_{α}^{eBay}, and R^{E} for seller revenue, respectively, in the eBay auction with risk-neutral bidders, in the eBay auction with CARA bidders, and in the English ascending bid auction. If the buy price B is between $\bar{B}$ and B_{α} then

$$R_{\alpha}^{eBay} > R^{eBay} = R^{E},$$

where the strict inequality follows by Corollary 3 and where the equality holds by Corollary 2. And, if the buy price B is between $\bar{B}$ and $\bar{v}$, then

$$R_{\alpha}^{Yahoo} > R^{Yahoo} = R^{E},$$

where the inequality holds by Corollary 4(ii) and the equality holds by Corollary 2. Thus both auctions yield more revenue than the ascending bid auction for a wide range of buy prices.

It turns out that the eBay and the Yahoo auction can be revenue ranked when bidders are risk averse. In particular, $R_{\alpha}^{Yahoo} > R_{\alpha}^{eBay}$ when the buy price is the same in both auctions, as we now show. Consider the case where $B \in (\bar{B}, B_{\alpha})$. From Figure 4 one can see that seller revenue is B in both auctions if $v_1 > c_{\alpha}$; seller revenue is v_2 in both auctions if either $v_1 < B$ or (v_1, v_2) lies below $t_{\alpha}(v_1)$. If $v_1 > B$ and (v_1, v_2) lies above $t_{\alpha}(v_1)$ but below the $v_1 = v_2$ line, then revenue differs between the auctions; it is B in the Yahoo auction, but in the eBay auction expected revenue is $E[v_2 | t_{\alpha}(v_1) \leq v_2 \leq v_1]$. This is less than B, since

$$E[v_2 | t_{\alpha}(v_1) \leq v_2 \leq v_1] < E[v_2 | 2B - v_1 \leq v_2 \leq v_1] = B.$$

An analogous argument shows that seller revenue is higher in the Yahoo auction than the eBay auction, whether $B > B_{\alpha}$ or $B < \bar{B}$, when bidders are risk averse. Thus, we have the following Proposition.

Figure 4 goes here.

Proposition 5: *Assume bidders are CARA risk averse with index of risk aversion $\alpha > 0$. Then expected seller revenue is higher in the Yahoo auction than in the eBay auction when the buy price is the same for both auctions.*

While a seller obtains more revenue with the Yahoo than the eBay auction, bidders are indifferent between the two auction formats when the buy price is the same.

Corollary 5: *Assume bidders are CARA risk averse with index of risk aversion $\alpha > 0$. The Yahoo and eBay auctions, each with buy price B, are payoff equivalent for the bidders; i.e., a bidder whose value is v obtains the same expected payoff in the Yahoo auction as in the eBay auction.*
Proof: Appendix.

Matthews (1987) shows that CARA bidders are indifferent between the English ascending bid auction and the first-price sealed-bid auction. If the buy price B is greater than B_α then it is never accepted in the eBay auction (Proposition 3(ii)) and CARA bidders are indifferent between the eBay auction and the ascending bid auction. Hence Corollary 5 and Matthews' result imply that, for $B > B_\alpha$, CARA bidders are indifferent between the eBay and Yahoo buy-now auctions, the first-price sealed-bid auction, and the English auction.

5 Conclusion

We have formulated models that capture key features of auctions with buy prices, as implemented on the eBay and Yahoo auction sites. In our analysis we characterize equilibrium strategies for risk neutral bidders in a simple environment with two bidders with symmetric, uniformly distributed independent private values. We show that the bidders' expected payoffs and expected seller revenue are the same for the eBay and Yahoo mechanisms. However, actual payments differ between the two mechanisms; a buy price that would be rejected by all bidder types in the eBay mechanism would be accepted by some types in the Yahoo mechanism. Under risk neutrality both types of buy-now auctions yield expected seller revenue that is less than or equal to that of the ascending bid auction. For the eBay buy–now auction, seller revenue is strictly less than revenue in the ascending bid auction if there is a positive probability that bidders accept the buy price. For the Yahoo buy-now auction, there is an interval of buy prices that are accepted by some bidder types and for which revenue equivalence with the ascending bid auction holds.

We derive several results for risk averse bidders, under the hypothesis of common CARA preferences. We characterize equilibria for the eBay and Yahoo buy-now auctions. For a given buy price, the buy price is more likely to be accepted when bidders are risk averse as when they are risk neutral. In addition, we show that for any amount of risk aversion, there is range of buy prices for each type of buy-now auction that yields greater expected seller revenue than the English ascending bid auction. Finally, we show that for a given buy price, expected seller revenue for the Yahoo auction exceeds that for the eBay auction, while expected bidder payoffs are the same in both auctions.

There are a number of ways in which this analysis might be extended. One could introduce time discounting, or a time-cost of waiting for bidders and for the seller. It has been suggested that waiting costs associated with Internet auctions make the buy price option attractive for bidders. Within the independent private values framework one could consider more general value distributions, larger (and perhaps, uncertain) numbers of bidders, and a reserve price. It would also be interesting to consider a common or affiliated values setting, since some goods auctioned on the Internet surely have a common value component.

6 Appendix

Proof of Proposition 1: If bidder i accepts the buy price, then his expected payoff when his rival uses the cutoff strategy c is

$$U^b(v_i, c) = \left[\frac{1}{2}(1 - F(c)) + F(c)\right] u(v_i - B) = \frac{(\frac{1}{2}\bar{v} + \frac{1}{2}c - \underline{v})}{\bar{v} - \underline{v}} u(v_i - B),$$

whereas if he waits then his expected payoff is

$$U^w(v_i, c) = \int_{\underline{v}}^{\min\{v_i, c\}} u(v_i - v_j) dF(v_j) = \frac{1}{\bar{v} - \underline{v}} \int_{\underline{v}}^{\min\{v_i, c\}} u(v_i - v_j) dv_j.$$

Clearly, a necessary condition for c to be an interior equilibrium cutoff value is that $U^b(c,c) = U^w(c,c)$.[10] Since bidders are risk neutral, this equality is equivalent to

$$\frac{1}{2}(\bar{v} + c - 2\underline{v})(c - B) = \int_{\underline{v}}^{c}(c - v_j)dv_j = \frac{1}{2}(c - \underline{v})^2.$$

Solving for c yields

$$c^* = \frac{B(\bar{v} - 2\underline{v}) + \underline{v}^2}{\bar{v} - B},$$

which is the value cut-off indicated in part (ii) of the proposition. Since $B > \underline{v}$ then $c^* > B$.

We now show for $B < \bar{B}$ that c^*, as given above, is an equilibrium cutoff value. It's easy to verify that $U^b(v_i, c^*)$ is linear in v_i, $U^w(v_i, c^*)$ is convex in v_i for $v_i < c^*$, and $U^w(v_i, c^*)$ is linear in v_i for $v_i > c^*$. Furthermore,

$$\left.\frac{\partial U^b(v_i, c^*)}{\partial v_i}\right|_{v_i = c^*} = \frac{1}{2}(1 + F(c^*)) > F(c^*) = \left.\frac{\partial U^w(v_i, c^*)}{\partial v_i}\right|_{v_i = c^*},$$

where the strict inequality holds since $B < \bar{B}$ implies $c^* < \bar{v}$ and hence $F(c^*) < 1$. This inequality means that the slope of the expected payoff function from accepting the buy price is steeper at $v_i = c^*$ than the expected payoff function from waiting. These facts imply that $U^w(v_i, c^*) > U^b(v_i, c^*)$ for $v_i < c^*$ and $U^w(v_i, c^*) < U^b(v_i, c^*)$ for $v_i > c^*$, i.e., c^* is a symmetric equilibrium. This establishes Proposition 1(ii).

If $B > \bar{B}$ then $c^* > \bar{v}$ (for c^* given above) and hence there is no interior equilibrium cutoff. We now show that B is not an equilibrium cutoff. Suppose it were. Since $\lim_{v_i \downarrow B} U^w(v_i, B) > 0$ and $\lim_{v_i \downarrow B} U^b(v_i, B) = 0$, there is some $v_i > B$ such that $U^w(v_i, B) > U^b(v_i, B)$, contradicting that B is an equilibrium cutoff. We now show that $\bar{v}$ is an equilibrium cutoff value. We have $U^b(v_i, \bar{v}) = v_i - B$ and

$$U^w(v_i, \bar{v}) = \int_{\underline{v}}^{v_i}(v_i - v_j)dF(v_j) = \frac{(v_i - \underline{v})^2}{2(\bar{v} - \underline{v})}.$$

[10] If $U^b(c,c) > U^w(c,c)$ then by the continuity of U^b and U^w we have $U^b(v', c) > U^w(v', c)$ for some $v' \in (B, c)$, which contradicts that c is an equilibrium cutoff. If $U^b(c,c) < U^w(c,c)$ then by the continuity of U^b and U^w we have $U^b(v', c) < U^w(v', c)$ for some $v' \in (c, \bar{v})$, again a contradiction.

It is straightforward to show that $U^w(v_i, \bar{v}) > U^b(v_i, \bar{v})$ for all $v_i < \bar{v}$. This establishes (i). $\square$

Proof of Proposition 2: We begin by writing the payoff function when the threshold function has a jump down. If a bidder's value is v, he chooses a threshold $\tilde{t}$, and every other bidder follows the threshold strategy t (which jumps down to $\underline{v}$ at z), then the bidder's expected utility is

$$U(\tilde{t}, v; t) = \begin{cases} \int_{\underline{v}}^{\tilde{t}} u(v-x)dF(x) + [F(B) - F(\tilde{t})]u(v-B) & \text{if } \tilde{t} > \bar{t} \\ \int_{\underline{v}}^{\tilde{t}} u(v-x)dF(x) + [F(t^{-1}(\tilde{t})) - F(\tilde{t})]u(v-B) & \text{if } \tilde{t} \in [\underline{t}, \bar{t}] \\ \int_{\underline{v}}^{\tilde{t}} u(v-x)dF(x) + [F(t^{-1}(\underline{t})) - F(\tilde{t})]u(v-B) & \text{if } \tilde{t} \in (\underline{v}, \underline{t}) \\ u(v-B)(F(z) + \frac{1}{2}(1-F(z))) & \text{if } \tilde{t} = \underline{v}. \end{cases}$$

The proof proceeds in several steps.

STEP I: We first show that a (symmetric) equilibrium threshold function satisfies $t(B) = B$. Suppose to the contrary that $t(B) < B$. Then

$$U(t(B), B; t) = \int_{\underline{v}}^{t(B)} u(B-x)dF(x) < \int_{\underline{v}}^{B} u(B-x)dF(x) = U(B, B; t),$$

where the inequality is strict since $u(B-x) > 0$ for $x \in [t(B), B)$ and since $dF(x) > 0$. In other words, a bidder with value $v = B$ obtains a higher payoff choosing a threshold of B than of $t(B)$ when the other bidder follows t. This contradicts that t is a symmetric equilibrium.

STEP II: Let t be an equilibrium threshold strategy which is differentiable (except at one point $\hat{v}$, if t jumps down at $\hat{v}$). We show that $t(v) = 2B - v$ for $v \in [B, \bar{v}]$ if t has no jump and $t(v) = 2B - v$ for $v \in [B, \hat{v})$ if t jumps down at $\hat{v}$.

Consider the case where t jumps down at $\hat{v}$. For $\tilde{v} \leq \hat{v}$ define

$$U(\tilde{v}, v) = \int_{\underline{v}}^{t(\tilde{v})} (v-x)\frac{1}{\bar{v}-\underline{v}}dx + (v-B)\frac{\tilde{v}-t(\tilde{v})}{\bar{v}-\underline{v}}$$

to be a bidder's expected payoff when his value is v, he chooses a threshold as though his value was $\tilde{v}$, and the other bidder follows t. For $v \in (B, \hat{v})$ we have $\underline{v} < t(v) < B$; hence a necessary condition for t to be an equilibrium is that $\frac{dU(\tilde{v},v)}{d\tilde{v}}|_{\tilde{v}=v} = 0$ for

$v \in (B, \hat{v})$, i.e.,

$$Bt' - tt' + v - B = 0.$$

Integrating this equation yields

$$-t^2 + 2Bt + v^2 - 2Bv = C,$$

where C is a constant. The boundary condition that $t(B) = B$ implies that $C = 0$, and thus

$$(v - t)(t - (2B - v)) = 0.$$

There are two solutions for the threshold strategy: (i) $t(v) = 2B - v$ and (ii) $t(v) = v$. The second solution, however, can be discarded since a threshold strategy must satisfy $t(v) \leq B$ but we have $v > B$. The proof for the case where t has no jump is straightforward and therefore omitted.

STEP III: Let t be an equilibrium threshold strategy which is differentiable (except at one point $\hat{v}$, if t jumps down at $\hat{v}$). We show that if $B \geq \bar{B}$ then $t(v) = 2B - v$ for $v \in [B, \bar{v}]$ and if $B < \bar{B}$ then t jumps down at $\hat{v} = \frac{\underline{v}^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B}$, i.e.,

$$t(v) = \begin{cases} 2B - v & \text{if } v \leq \frac{\underline{v}^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B} \\ \underline{v} & \text{otherwise.} \end{cases}$$

Suppose that t jumps down at z. Then for $v \leq z$ we have

$$U(v, v) = \int_{\underline{v}}^{2B-v} (v - x)\frac{1}{\bar{v} - \underline{v}}dx + (v - B)\frac{v - [2B - v]}{\bar{v} - \underline{v}} = \frac{1}{2}\frac{(v - \underline{v})^2}{\bar{v} - \underline{v}}.$$

Define $U^*(v)$ to be a bidder's payoff if his value is v, he chooses a threshold $\underline{v}$ and his rival follows $t(v) = 2B - v$. Then

$$U^*(v) = \frac{z - \underline{v}}{\bar{v} - \underline{v}}(v - B) + \frac{1}{2}\frac{\bar{v} - z}{\bar{v} - \underline{v}}(v - B).$$

A necessary condition for t to jump down at z is that a bidder is indifferent between the thresholds $2B - z$ and $\underline{v}$ when his value is z, i.e., that $U(z, z) = U^*(z)$. This implies $z = \frac{\underline{v}^2 + B(\bar{v} - 2\underline{v})}{\bar{v} - B}$. It's easy to see that $z > B$ for $B > \underline{v}$. If $B > \bar{B}$ then $z > \bar{v}$, which contradicts that t jumps down for some $z \in (B, \bar{v})$. If $B < \bar{B}$ then $z \in (B, \bar{v})$.

STEP IV: We have shown that if t is an equilibrium in threshold strategies which are continuous and differentiable (except at possibly one point), then t is as given in Proposition 2. We now show the threshold strategy given in Proposition 2 is indeed an equilibrium, and therefore an equilibrium exists.

Case (i): Assume $B > \bar{B}$. Given the threshold strategy in Prop. 2(i), a bidder whose value is v has a best response in $[t(\bar{v}), B]$. The bidder's payoff, if he chooses a threshold as though his true value were $\tilde{v} \in [B, \bar{v}]$, is

$$U(\tilde{v}, v) = \int_{\underline{v}}^{2B-\tilde{v}} (v - x) \frac{1}{\bar{v} - \underline{v}} dx + (v - B) \int_{2B-\tilde{v}}^{\tilde{v}} \frac{1}{\bar{v} - \underline{v}} dx.$$

Note that

$$\frac{dU(\tilde{v}, v)}{d\tilde{v}} = \frac{1}{\bar{v} - \underline{v}} (v - \tilde{v}),$$

and

$$\frac{d^2U(\tilde{v}, v)}{d\tilde{v}^2} = \frac{-1}{\bar{v} - \underline{v}} < 0.$$

Hence $U(\tilde{v}, v)$ is concave in $\tilde{v}$ (for fixed v), and $\tilde{v} = v$ maximizes $U(\tilde{v}, v)$ on $[B, \bar{v}]$.

Case (ii): Assume $B < \bar{B}$. Given the threshold strategy in Prop. 2(ii), a bidder's payoff function is discontinuous at a threshold of $\underline{v}$. Hence we need to show that (a) $U(v, v) > U^*(v)$ if $v \in [B, \hat{v})$ and (b) $U(\hat{v}, v) < U^*(v)$ if $v \in (\hat{v}, \bar{v}]$. To establish (a) note that

$$\left.\frac{dU(v, v)}{dv}\right|_{v=\hat{v}} = \frac{\hat{v} - \underline{v}}{\bar{v} - \underline{v}} < \frac{\hat{v} - \underline{v}}{\bar{v} - \underline{v}} + \frac{1}{2} \frac{\bar{v} - \hat{v}}{\bar{v} - \underline{v}} = \frac{dU^*(v)}{dv}.$$

In other words, at $v = \hat{v}$ we have that $U^*(v)$ is steeper than $U(v, v)$. Since $U(v, v)$ is strictly convex in v, $U^*(v)$ is linear in v, and $U(\hat{v}, \hat{v}) = U^*(\hat{v})$ then (a) holds. For $v \in [\hat{v}, \bar{v}]$ we have that $U(\hat{v}, v)$ and $U^*(v)$ are both linear in v, with

$$\frac{dU(\hat{v}, v)}{dv} = \frac{\hat{v} - \underline{v}}{\bar{v} - \underline{v}} < \frac{\hat{v} - \underline{v}}{\bar{v} - \underline{v}} + \frac{1}{2} \frac{\bar{v} - \hat{v}}{\bar{v} - \underline{v}} = \frac{dU^*(v)}{dv}.$$

Since $U(\hat{v}, \hat{v}) = U^*(\hat{v})$ this establishes (b). $\square$

Proof of Corollary 2: By Corollary 1 revenue is the same in the eBay and the Yahoo auction; hence to compare the revenues it is sufficient to compare the revenue of the eBay and the English ascending bid auction. If $B > \bar{B}$ then the buy price is never accepted in the eBay auction (Proposition 1) and hence expected revenue is trivially the same as in the English auction.

Suppose $B < \bar{B}$. If both bidders have a value less than or equal to c^* then revenue is the same in both auctions (it is $\min(v_1, v_2)$). If one or both bidders has a value above c^* then the seller receives B in the eBay auction and receives $\min(v_1, v_2)$ in the English auction. Hence, the difference Δ in revenue is

$$\Delta = 2\int_{c^*}^{\bar{v}} \left[\int_{\underline{v}}^{v_1} (B - v_2)dF(v_2)\right] dF(v_1).$$

Integrating the right-hand side yields

$$\Delta = \frac{2}{(\bar{v} - \underline{v})^2}[\frac{1}{2}B\bar{v}^2 - \frac{1}{2}Bc^{*2} - \frac{1}{6}\bar{v}^3 + \frac{1}{6}c^{*3} - B\bar{v}\underline{v} + Bc^*\underline{v} + \frac{1}{2}\bar{v}\underline{v}^2 - \frac{1}{2}c^*\underline{v}^2],$$

or

$$\Delta = \frac{(\bar{v} - c^*)}{(\bar{v} - \underline{v})^2}[Bc^* + \{B(\bar{v} - 2\underline{v}) + \underline{v}^2\} - \frac{1}{3}(\bar{v}^2 + \bar{v}c^* + c^{*2})].$$

The expression in curly brackets above is equal to $(\bar{v}-B)c^*$, according to the definition of c^* in (1). Substituting and simplifying yields

$$\Delta = -\frac{(\bar{v} - c^*)^3}{3(\bar{v} - \underline{v})^2}.$$

Hence $\Delta < 0$ since $B < \bar{B}$ implies $c^* < \bar{v}$. $\square$

Proof of Lemma 1: (i) Equation (3), after replacing c_α with c to reduce notation, can be rewritten as

$$e^{\alpha B} = \frac{(\alpha\bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha\underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})}.$$

Solving for the buy price B as a function of the cutoff value c yields

$$B = \frac{1}{\alpha}\ln\left[\frac{(\alpha\bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha\underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})}\right].$$

Let $\lambda(c)$ be the function inside the square brackets and define

$$\mu(c) = \frac{1}{\alpha}\ln\lambda(c) = \frac{1}{\alpha}\ln\left[\frac{(\alpha\bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha \underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})}\right].$$

Then $B = \mu(c)$, i.e., $\mu(c)$ gives for each c the value of B such that equation (3) holds. Clearly $\mu(\underline{v}) = \underline{v}$. We will show that (a) $\mu'(c) > 0$ and (b) $\mu(c) < c$ for $c > \underline{v}$. Clearly $B_\alpha = \mu(\bar{v})$, where B_α is defined in (2). These facts imply $\mu^{-1}(B)$ is well-defined on $[\underline{v}, B_\alpha]$, i.e., for each $B \in [\underline{v}, B_\alpha]$ there is a unique $c \in [B, \bar{v}]$ satisfying equation (3). Furthermore, $\mu^{-1}(B)$ is strictly increasing in B, $\mu^{-1}(\underline{v}) = \underline{v}$, and $\mu^{-1}(B_\alpha) = \bar{v}$. We complete the proof by establishing (a) and (b).

Proof of (a): To see that μ is strictly increasing in c observe that

$$\mu'(c) = \frac{1}{\alpha}\frac{\lambda'(c)}{\lambda(c)},$$

where

$$\lambda'(c) = \frac{e^{\alpha c}\left\{\alpha(\bar{v}^2 - c^2) + 2c - 2\underline{v}(1 + \alpha\bar{v} - \alpha c) - \frac{2}{\alpha}\right\} + \frac{2}{\alpha}e^{\alpha\underline{v}}}{(\bar{v} + c - 2\underline{v})^2}.$$

Clearly $\lambda(c) > 0$ for $c \in [\underline{v}, \bar{v}]$. To see that $\lambda'(c) > 0$ also, define $g(c)$ as

$$g(c) = e^{\alpha c}\left\{\alpha(\bar{v}^2 - c^2) + 2c - 2\underline{v}(1 + \alpha\bar{v} - \alpha c) - \frac{2}{\alpha}\right\},$$

so that

$$\lambda'(c) = \frac{g(c) + \frac{2}{\alpha}e^{\alpha\underline{v}}}{(\bar{v} + c - 2\underline{v})^2}.$$

Since $g'(c) = e^{\alpha c}\alpha^2(\bar{v} - c)(\bar{v} + c - 2\underline{v}) \geq 0$, then

$$g(c) \geq g(\underline{v}) = e^{\alpha\underline{v}}\left\{\alpha(\bar{v}^2 - \underline{v}^2) - 2\underline{v}\alpha(\bar{v} - \underline{v}) - \frac{2}{\alpha}\right\},$$

and hence

$$\lambda'(c) \geq \frac{e^{\alpha\underline{v}}\left\{\alpha(\bar{v}^2 - \underline{v}^2) - 2\underline{v}\alpha(\bar{v} - \underline{v}) - \frac{2}{\alpha}\right\} + \frac{2}{\alpha}e^{\alpha\underline{v}}}{(\bar{v} + c - 2\underline{v})^2} = \frac{e^{\alpha\underline{v}}\alpha(\bar{v} - \underline{v})^2}{(\bar{v} + c - 2\underline{v})^2} > 0,$$

for $c \in [\underline{v}, \bar{v}]$. This establishes (a).

Proof of (b): We show that $\mu(c) < c$ for $c > \underline{v}$, i.e.,

$$\mu(c) = \frac{1}{\alpha} \ln \left[\frac{(\alpha\bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha \underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})} \right] < c.$$

Equivalently, we need to establish that

$$\frac{(\alpha\bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha \underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})} < e^{\alpha c}.$$

Rearranging this expression yields

$$-\alpha(c - \bar{v}) + 1 < e^{-\alpha(c-\bar{v})}.$$

However, this inequality holds for $c > \underline{v}$ since $e^x > x + 1$ for $x \neq 0$.[11] In particular, choosing $x = -\alpha(c - \underline{v})$ yields the result. This establishes (b).

Proof of (ii): We show that if $B < B_\alpha$ then $c_\alpha < c^*$. By Proposition 1, when bidders are risk neutral and $B \in (\underline{v}, \bar{B})$, then the unique symmetric equilibrium cutoff value is

$$c^* = \frac{B(\bar{v} - 2\underline{v}) + \underline{v}^2}{\bar{v} - B}.$$

It will be convenient to express this in inverse form:

$$B = \frac{c^*\bar{v} - \underline{v}^2}{\bar{v} + c^* - 2\underline{v}}.$$

Defining $n(c)$ as

$$n(c) \equiv e^{\frac{\alpha(c\bar{v} - \underline{v}^2)}{\bar{v} + c - 2\underline{v}}},$$

then $e^{\alpha B} = n(c^*)$. With risk-averse bidders and $B < B_\alpha$, the symmetric equilibrium cutoff value c_α satisfies

$$e^{\alpha B} = \frac{(\alpha\bar{v} - \alpha c_\alpha + 2)e^{\alpha c_\alpha} - 2e^{\alpha \underline{v}}}{\alpha(\bar{v} + c_\alpha - 2\underline{v})}.$$

[11] Let $f(x) = e^x - (x + 1)$. Since $f'(x) = e^x - 1$ and $f''(x) = e^x > 0$, this means that f has a global minimum at $x = 0$ where $f(0) = 0$.

(See the proof of Lemma 1(i).) Defining $\lambda(c)$ as

$$\lambda(c) \equiv \frac{(\alpha\bar{v} - \alpha c + 2)e^{\alpha c} - 2e^{\alpha \underline{v}}}{\alpha(\bar{v} + c - 2\underline{v})},$$

then $e^{\alpha B} = \lambda(c_\alpha)$. By Lemma 1(i) we have $c_\alpha < \bar{v}$. Hence to establish that $c_\alpha < c^*$ it is sufficient to show that $\lambda(c) > n(c)$ for $c \in (\underline{v}, \bar{v}]$.

Let $\hat{\lambda}(c) = \alpha(\bar{v} + c - 2\underline{v})\lambda(c)$ and $\hat{n}(c) = \alpha(\bar{v} + c - 2\underline{v})n(c)$. The inequality $\lambda(c) > n(c)$ is equivalent to $\hat{\lambda}(c) > \hat{n}(c)$. Note that $\hat{\lambda}(\underline{v}) = \hat{n}(\underline{v}) = \alpha(\bar{v} - \underline{v})e^{\alpha\underline{v}}$. The derivatives of $\hat{\lambda}(c)$ and $\hat{n}(c)$ are

$$\hat{\lambda}'(c) = e^{\alpha c}[\alpha^2(\bar{v} - c) + \alpha],$$

and

$$\hat{n}'(c) = e^{\frac{\alpha(c\bar{v} - \underline{v}^2)}{\bar{v} + c - 2\underline{v}}} \left[\alpha + \frac{\alpha^2(\bar{v} - \underline{v})^2}{\bar{v} + c - 2\underline{v}}\right].$$

We will show that $\hat{\lambda}'(c) > \hat{n}'(c)$ by establishing several claims that lead to this result. Let $z = \bar{v} - \underline{v} > 0$ and let $y = c - \underline{v} \in [0, z]$.

Claim #1:

$$\frac{\alpha y^2}{y + z} + 1 = \frac{y + z + \alpha y^2}{y + z} \geq \frac{y + z + \alpha z^2}{(y + z)(1 + \alpha(z - y))}$$

with strict inequality for $y < z$. This claim follows from algebraic manipulation.

Claim #2:

$$e^{\frac{\alpha y^2}{y+z}} \geq \frac{\alpha y^2}{y + z} + 1$$

with strict inequality for $y > 0$. This claim follows from the properties of the exponential function.

Claim #3:

$$e^{\frac{\alpha y^2}{y+z}} > \frac{y + z + \alpha z^2}{(y + z)(1 + \alpha(z - y))}$$

for $y = c - \underline{v} \in [0, z]$. This claim follows from claims 1 and 2.

Using the definitions of y and z, Claim #3 may be expressed as

$$e^{\frac{\alpha(c - \underline{v})^2}{\bar{v} + c - 2\underline{v}}} > \frac{\bar{v} + c - 2\underline{v} + \alpha(\bar{v} - \underline{v})^2}{(\bar{v} + c - 2\underline{v})(1 + \alpha(\bar{v} - c))}.$$

This inequality implies that

$$\hat{\lambda}'(c) = e^{\alpha c}[\alpha^2(\bar{v} - c) + \alpha] > e^{\frac{\alpha(c\bar{v} - \underline{v}^2)}{\bar{v}+c-2\underline{v}}}[\alpha + \frac{\alpha^2(\bar{v} - \underline{v})^2}{\bar{v} + c - 2\underline{v}}] = \hat{n}'(c),$$

which is the desired result. $\square$

Proof of Proposition 3: If bidder i accepts the buy price then his expected utility, when his rival uses the cutoff strategy c, is

$$U^b(v_i, c) = \left[\frac{1}{2}(1 - F(c)) + F(c)\right] u(v_i - B) = \frac{\bar{v} + c - 2\underline{v}}{2(\bar{v} - \underline{v})}[1 - e^{-\alpha(v_i - B)}],$$

whereas if he waits then his expected utility is

$$\begin{aligned} U^w(v_i, c) &= \int_{\underline{v}}^{\min\{v_i, c\}} u(v_i - v_j) dF(v_j) \\ &= \frac{\min\{v_i, c\} - \underline{v} - \frac{e^{-\alpha v_i}}{\alpha}(e^{\alpha \min\{v_i, c\}} - e^{\alpha \underline{v}})}{\bar{v} - \underline{v}}. \end{aligned}$$

We first establish that $B > \underline{v}$ implies that $c = B$ is not a symmetric boundary equilibrium. Clearly $U^b(B, B) = 0$, and since $B > \underline{v}$ then $U^w(B, B) > 0$. Since U^b and U^w are continuous in v_i then there is an $\varepsilon > 0$ such that $U^w(v_i, B) > U^b(v_i, B)$ for $v_i \in [B, B + \varepsilon]$. Hence $c = B$ is not a symmetric equilibrium.

Proof of (i): Assume $B \in (\underline{v}, B_\alpha)$. A necessary condition for c to be a symmetric interior (i.e., $B < c < \bar{v}$) equilibrium cutoff value is that $U^b(c, c) = U^w(c, c)$, i.e.,

$$\frac{\bar{v} + c - 2\underline{v}}{2(\bar{v} - \underline{v})}[1 - e^{-\alpha(c - B)}] = \frac{c - \underline{v} - \frac{e^{-\alpha c}}{\alpha}(e^{\alpha c} - e^{\alpha \underline{v}})}{\bar{v} - \underline{v}}.$$

By Lemma 1 there is a unique c satisfying this equation, and hence there is (at most) one symmetric equilibrium. Moreover, $B < c < \bar{v}$.

We show that c is a symmetric equilibrium in cutoff strategies, i.e., if $v_i \in [B, c)$ then $U^w(v_i, c) > U^b(v_i, c)$ and if $v_i \in (c, \bar{v}]$ then $U^b(v_i, c) > U^w(v_i, c)$. We establish this by showing that $U^b(v_i, c)$ is everywhere steeper than $U^w(v_i, c)$. Hence they cross at $v_i = c$, with U^b below U^w for $v_i \in [\underline{v}, c)$ and U^b above U^w for $v_i \in (c, \bar{v}]$. We have

$$\frac{\partial U^b(v_i, c)}{\partial v_i} = \frac{\bar{v} + c - 2\underline{v}}{2(\bar{v} - \underline{v})} \alpha e^{-\alpha(v_i - B)},$$

and

$$\frac{\partial U^w(v_i,c)}{\partial v_i} = \begin{cases} \frac{1}{\bar{v}-\underline{v}}[1-e^{-\alpha(v_i-\underline{v})}] & \text{if } v_i \le c \\ \\ \frac{1}{\bar{v}-\underline{v}}[e^{-\alpha(v_i-c)}-e^{-\alpha(v_i-\underline{v})}] & \text{if } v_i > c. \end{cases}$$

Note that $U^w(v_i,c)$ is differentiable at $v_i = c$. Since c solves (3) then

$$e^{\alpha B} = \frac{(\alpha\bar{v}-\alpha c+2)e^{\alpha c}-2e^{\alpha\underline{v}}}{\alpha(\bar{v}+c-2\underline{v})} > \frac{2e^{\alpha c}-2e^{\alpha\underline{v}}}{\alpha(\bar{v}+c-2\underline{v})},$$

where the inequality follows from $\bar{v} > c$. Rearranging yields

$$\frac{\alpha(\bar{v}+c-2\underline{v})}{2(\bar{v}-\underline{v})}e^{\alpha B} > \frac{e^{\alpha c}-e^{\alpha\underline{v}}}{\bar{v}-\underline{v}},$$

which implies for $v_i \in [c,\bar{v})$ that

$$\frac{\partial U^b(v_i,c)}{\partial v_i} = \frac{\bar{v}+c-2\underline{v}}{2(\bar{v}-\underline{v})}\alpha e^{-\alpha(v_i-B)} > \frac{e^{-\alpha(v_i-c)}-e^{-\alpha(v_i-\underline{v})}}{\bar{v}-\underline{v}} = \frac{\partial U^w(v_i,c)}{\partial v_i}.$$

In particular, for $v_i = c$ we have

$$\left.\frac{\partial U^b(v_i,c)}{\partial v_i}\right|_{v_i=c} > \left.\frac{\partial U^w(v_i,c)}{\partial v_i}\right|_{v_i=c}.$$

For $v_i < c$, then $U^b(v_i,c)$ is strictly concave in v_i and $U^w(v_i,c)$ is strictly convex in v_i. Hence this same inequality holds for $v_i < c$. This establishes that c is the unique interior symmetric equilibrium.

We have already shown that $c = \underline{v}$ is not a symmetric boundary equilibrium. To see that $c = \bar{v}$ is not a symmetric equilibrium either, note that by Lemma 1 we have $U^b(\bar{v},\bar{v}) = U^w(\bar{v},\bar{v})$ if $B = B_\alpha$. Since

$$\frac{dU^b(\bar{v},\bar{v})}{dB} = -\alpha e^{-\alpha(\bar{v}-B)} < 0$$

and $\frac{dU^w(\bar{v},\bar{v})}{dB} = 0$, then $B < B_\alpha$ implies $U^b(\bar{v},\bar{v}) > U^w(\bar{v},\bar{v})$. The continuity of U^b and U^w imply there is an $\varepsilon > 0$ such that $U^b(v_i,\bar{v}) > U^w(v_i,\bar{v})$ $\forall v_i \in (\bar{v}-\varepsilon,\bar{v}]$. Thus, $\bar{v}$ is not a symmetric equilibrium. Hence there is no symmetric boundary equilibrium.

We now prove Proposition 3(ii). Suppose that $B \ge B_\alpha$. Then

$$e^{\alpha B} \ge e^{\alpha B_\alpha} = \frac{e^{\alpha\bar{v}}-e^{\alpha\underline{v}}}{\alpha(\bar{v}-\underline{v})}. \tag{5}$$

Assume $v_i < \bar{v}$ and define $x = \alpha(\bar{v} - v_i)$. Then

$$e^{\alpha\bar{v}} = e^{\alpha v_i} e^x > e^{\alpha v_i}(x+1) = e^{\alpha v_i}(\alpha(\bar{v} - v_i) + 1), \tag{6}$$

where the inequality follows since $e^x > x + 1$ for $x \neq 0$. Combining inequalities (5) and (6) yields

$$e^{\alpha B} > \frac{e^{\alpha v_i}(\alpha(\bar{v} - v_i) + 1) - e^{\alpha \underline{v}}}{\alpha(\bar{v} - \underline{v})},$$

or

$$e^{-\alpha(v_i - B)} > \frac{\alpha(\bar{v} - v_i) + 1 - e^{-\alpha(v_i - \underline{v})}}{\alpha(\bar{v} - \underline{v})}.$$

This implies

$$U^b(v_i, \bar{v}) = 1 - e^{-\alpha(v_i - B)} < \frac{1}{(\bar{v} - \underline{v})}[v_i - \underline{v} - \frac{1}{\alpha} + \frac{1}{\alpha} e^{-\alpha(v_i - \underline{v})}] = U^w(v_i, \bar{v}).$$

Hence $U^b(v_i, \bar{v}) < U^w(v_i, \bar{v})$ for all $v_i \in [B, \bar{v})$, i.e., $c = \bar{v}$ is a symmetric equilibrium. □

Proof of Proposition 4: The proof proceeds in two steps. Step I shows that the necessary conditions for a symmetric equilibrium uniquely identify the given threshold strategy. Step II shows that the given threshold strategy is, in fact, an equilibrium.

STEP I: Let $t(v)$ be a symmetric equilibrium, possibly with a jump down at $\hat{v} \in (B, \bar{v})$. A bidder's expected utility if his value is v, he chooses a threshold $t(\tilde{v}) > \underline{v}$, and his rival follows t is

$$U(\tilde{v}, v) = \int_{\underline{v}}^{t(\tilde{v})} (1 - e^{-\alpha(v-x)}) \frac{1}{\bar{v} - \underline{v}} dx + \int_{t(\tilde{v})}^{\tilde{v}} (1 - e^{-\alpha(v-B)}) \frac{1}{\bar{v} - \underline{v}} dx.$$

A necessary condition for t to be a symmetric interior equilibrium is that $\frac{\partial U(\tilde{v}, v)}{\partial \tilde{v}}|_{\tilde{v}=v} = 0$ for $v \in [B, \hat{v})$, i.e.,

$$t'(v)[1 - e^{-\alpha(v - t(v))}] + (1 - t'(v))[1 - e^{-\alpha(v-B)}] = 0,$$

which can be rewritten as

$$t'(v) = -\frac{1 - e^{-\alpha(v-B)}}{e^{-\alpha(v-B)} - e^{-\alpha(v - t(v))}}.$$

Integrating both sides and using the initial condition that $t(B) = B$ to set the constant of integration to zero, we find that $t(v)$ is defined implicitly by

$$A(t) \equiv te^{\alpha B} - \frac{1}{\alpha}e^{\alpha t} = -\frac{1}{\alpha}e^{\alpha v} + ve^{\alpha B} \equiv D(v). \tag{7}$$

$A(t)$ is continuous, increasing in t for $t < B$, decreasing in t for $t > B$, and attains a maximum at $t = B$. For $v \in [B, \bar{v}]$ clearly $D(v)$ is continuous and strictly decreasing. Figure 5 graphs $A(t)$. Note that $A(B) = D(B)$. Since $t = B$ is the unique value for which A attains its maximum of $A(B)$, then $t = B$ is the unique solution to (7) if $v = B$. For a given value $v' \in (B, \bar{v}]$ there are two values of t which solve $A(t) = D(v')$. The first solution is $t = v'$; graphically, this solution corresponds to the value of t where $A(t) = D(v')$ and A is decreasing. A second solution corresponds to the value of t where $A(t) = D(v')$ and A is increasing. Note that this second solution is less than the buy price B since $t < B$ for the domain in which A is increasing.

Figure 5 goes here.

We discard the first solution since a threshold strategy cannot have a threshold above the buy price. For each $v \in (B, \bar{v}]$ denote by $s(v)$ the unique $t < B$ satisfying (7). If $t(v)$ is a symmetric equilibrium then $t(v)$ must satisfy the first order necessary condition $t(v) = s(v)$ whenever the threshold is interior; i.e., whenever $t(v) > \underline{v}$.

Case (i): Suppose $B \geq B_\alpha$ where B_α is defined in (2). Then $D(\bar{v}) \geq A(\underline{v})$ and hence $s(v) > \underline{v}$ for $v \in [B, \bar{v})$. Hence, in the class of threshold strategies without a jump down there is at most one symmetric equilibrium. We now show there is no symmetric equilibrium with a jump down and hence a symmetric equilibrium in threshold strategies (if one exists) is unique. Suppose to the contrary that $t(v)$ is a symmetric equilibrium threshold strategy with a jump down at some value $\hat{v} < \bar{v}$. Then $t(v) = s(v)$ for $v \in [B, \hat{v}]$ and $t(v) = \underline{v}$ for $v \in (\hat{v}, \bar{v}]$. Then for $v \in [B, \hat{v}]$ the equilibrium expected payoff of a bidder whose value is v is

$$U(v, v) = \int_{\underline{v}}^{t(v)} (1 - e^{-\alpha(v-x)}) \frac{1}{\bar{v} - \underline{v}} dx + \int_{t(v)}^{v} (1 - e^{-\alpha(v-B)}) \frac{1}{\bar{v} - \underline{v}} dx. \tag{8}$$

Simplifying yields

$$U(v, v) = \frac{1}{\bar{v} - \underline{v}} \left(v - \underline{v} + e^{-\alpha v} \left[t(v)e^{\alpha B} - \frac{1}{\alpha}e^{\alpha t(v)} + \frac{1}{\alpha}e^{\alpha \underline{v}} - ve^{\alpha B} \right] \right).$$

Since $t(v)$ satisfies (7), then for $v \in [B, \hat{v}]$ we obtain

$$U(v,v) = \frac{1}{\bar{v} - \underline{v}} \left(v - \underline{v} - \frac{1}{\alpha} + \frac{1}{\alpha} e^{-\alpha(v - \underline{v})} \right).$$

Denote by $U^*(v)$ a bidder's payoff if his value is v, he chooses a threshold $\underline{v}$ and his rival follows $t(v)$. Then

$$U^*(v) = \left[F(\hat{v}) + \frac{1}{2}(1 - F(\hat{v})) \right] (1 - e^{-\alpha(v-B)}) = \frac{\bar{v} + \hat{v} - 2\underline{v}}{2(\bar{v} - \underline{v})}(1 - e^{-\alpha(v-B)}). \qquad (9)$$

A necessary condition for $t(v)$ to be a symmetric equilibrium with a jump down at $\hat{v} < \bar{v}$ is that $U(\hat{v}, \hat{v}) = U^*(\hat{v})$. This equality is equivalent to equation (3) in Lemma 1 which implicitly defines c_α. Since $B \geq B_\alpha$, by Lemma 1 there is no such $\hat{v} \in [B, \bar{v})$; hence there is no symmetric equilibrium with a jump down.

Case (ii) Suppose $B < B_\alpha$. Then $s(\overline{v}) < \underline{v}$. Let v_0 be such that $s(v_0) = \underline{v}$. The value v_0 satisfies $A(\underline{v}) = D(v_0)$, i.e.,

$$e^{\alpha B} = \frac{e^{\alpha v_0} - e^{\alpha \underline{v}}}{\alpha(v_0 - \underline{v})}.$$

Clearly $v_0 < \bar{v}$. We now show that the symmetric equilibrium threshold function must have a jump down. Suppose to the contrary that $t(v)$ is a symmetric equilibrium without a jump down. Then $t(v) = s(v)$ for $v \in [B, v_0]$ and $t(v) = \underline{v}$ for $v \in (v_0, \bar{v}]$.

Using $t(v_0) = \underline{v}$ and (8) we have that

$$U(v_0, v_0) = \int_{\underline{v}}^{v_0} (1 - e^{-\alpha(v-B)}) \frac{1}{\bar{v} - \underline{v}} dx = \frac{v_0 - \underline{v}}{\bar{v} - \underline{v}}(1 - e^{-\alpha(v_0 - B)}).$$

(In this case a bidder's equilibrium payoff function is discontinuous at $v = v_0$, so $U(v,v)$ is his equilibrium payoff only when his value v is less than v_0.) By (9) we have

$$U^*(v_0) = \frac{\bar{v} + v_0 - 2\underline{v}}{2(\bar{v} - \underline{v})}(1 - e^{-\alpha(v_0 - B)}).$$

Clearly $v_0 < \bar{v}$ implies $U(v_0, v_0) < U^*(v_0)$. Since $U(v,v)$ and $U^*(v)$ are both continuous on $[B, v_0]$ there is a $v < v_0$ such that $U(v,v) < U^*(v)$, i.e., a bidder whose value is v obtains a higher payoff with a threshold $\underline{v}$ than he obtains from his equilibrium threshold $t(v)$, contradicting that $t(v)$ is an equilibrium.

We have established that if $B < B_\alpha$ then a symmetric equilibrium must have a jump down. Clearly, the jump down must occur before $s(v)$ reaches $\underline{v}$, i.e., at some value $\hat{v} < v_0$. Suppose $t(v)$ is a symmetric equilibrium threshold strategy with a jump down at $\hat{v} < v_0$. Since $B < B_\alpha$, by Lemma 1 there is a unique $\hat{v}$ satisfying the necessary condition that at a jump down $U(\hat{v}, \hat{v}) = U^*(\hat{v})$.

We now derive the closed form for this threshold function. (7) can be rewritten as

$$\alpha t(v) - e^{\alpha(t(v)-B)} = \alpha v - e^{\alpha(v-B)}.$$

Rearranging terms, yields

$$we^w = -e^{\alpha(v-B)-e^{\alpha(v-B)}},$$

where $w = -e^{\alpha(t(v)-B)}$ and where $0 > -e^{\alpha(v-B)-e^{\alpha(v-B)}} \geq -e^{-1}$. This equation has two solutions, corresponding to the first and second solution in Figure 5. The solution satisfying $t(v) \leq B$ is given by

$$w = -e^{\alpha(t(v)-B)} = L^W\left[-e^{\alpha(v-B)-e^{\alpha(v-B)}}\right],$$

where L^W is the LambertW function. Solving for $t(v)$ yields

$$t(v) = B + \frac{1}{\alpha}\ln\left(-L^W\left[-e^{\alpha(v-B)-e^{\alpha(v-B)}}\right]\right),$$

which is the threshold strategy given in (4).

Step II: (Sufficiency) Case (i): Assume $B \geq B_\alpha$. Given the threshold strategy in Prop. 4(i), a bidder whose value is v has a best response in $[t(\bar{v}), B]$. We show that for each of a bidder's possible values v, the objective $U(\tilde{v}, v)$ is concave in $\tilde{v}$, and hence the first order condition is sufficient. We have that

$$\frac{dU(\tilde{v}, v)}{d\tilde{v}} = \frac{1}{\bar{v} - \underline{v}}\left\{t'(\tilde{v})[1 - e^{-\alpha(v-t(\tilde{v}))}] + (1 - t'(\tilde{v}))[1 - e^{-\alpha(v-B)}]\right\},$$

and

$$\frac{d^2U(\tilde{v}, v)}{d\tilde{v}^2} = \frac{1}{\bar{v} - \underline{v}}\left\{-[t'(\tilde{v})]^2\alpha e^{-\alpha(v-t'(\tilde{v}))} + t''(\tilde{v})[e^{-\alpha(v-B)} - e^{-\alpha(v-t(\tilde{v}))}]\right\}.$$

Clearly the first term in this expression is negative and the term $e^{-\alpha(v-B)} - e^{-\alpha(v-t(\tilde{v}))}$ is strictly positive since $t(\tilde{v}) < B$. Hence to establish that $U(\tilde{v}, v)$ is concave in $\tilde{v}$ it is sufficient to establish that $t''(\tilde{v}) < 0$. For compactness of notation, define $\Psi(v) \equiv L^W(-e^{\alpha(v-B)-e^{\alpha(v-B)}})$. We have

$$t'(\tilde{v}) = -\frac{e^{\alpha(\tilde{v}-B)} - 1}{1 + \Psi(\tilde{v})}.$$

Since both $1 + \Psi(\tilde{v}) > 0$ and $e^{\alpha(\tilde{v}-B)} - 1 > 0$ for $\tilde{v} > B$, then $t'(\tilde{v}) < 0$. Note that

$$\frac{d\Psi(v)}{dv} = -\alpha(e^{\alpha(v-B)} - 1)\frac{\Psi(v)}{1 + \Psi(v)}.$$

Hence

$$t''(\tilde{v}) = -\frac{(1 + \Psi(\tilde{v}))\alpha e^{\alpha(\tilde{v}-B)} - (e^{\alpha(\tilde{v}-B)} - 1)\left(-\alpha(e^{\alpha(\tilde{v}-B)} - 1)\frac{\Psi(\tilde{v})}{1+\Psi(\tilde{v})}\right)}{(1 + \Psi(\tilde{v}))^3}.$$

Rearranging, we obtain

$$t''(\tilde{v}) = -\alpha\frac{(\Psi(\tilde{v}) + e^{\alpha(\tilde{v}-B)})(e^{\alpha(\tilde{v}-B)}\Psi(\tilde{v}) + 1)}{(1 + \Psi(\tilde{v}))^4}.$$

For $\tilde{v} > B$ both $\Psi(\tilde{v}) + e^{\alpha(\tilde{v}-B)} > 0$ and $e^{\alpha(\tilde{v}-B)}\Psi(\tilde{v}) + 1 > 0$, and hence $t''(\tilde{v}) < 0$.

Case (ii). Assume $B < B_\alpha$. Given the threshold strategy in Prop. 4(ii), a bidder's payoff function is discontinuous at a threshold of $\underline{v}$. Since $U(\tilde{v}, v)$ is concave in $\tilde{v}$ a bidder whose value is v either has a best response of $t(v)$ if $v \leq \hat{v}$ (and $t(\hat{v})$ if $v > \hat{v}$) or has a best response of $\underline{v}$. Hence we need to show that (i) $U(v, v) > U^*(v)$ if $v \in [B, \hat{v})$ and (ii) $U(\hat{v}, v) < U^*(v)$ if $v \in (\hat{v}, \bar{v}]$.

For $v \in [B, \hat{v}]$ we have

$$\frac{dU(v, v)}{dv} = \frac{1}{\bar{v} - \underline{v}}\left(1 - e^{-\alpha(v-\underline{v})}\right).$$

For $v \in [B, \bar{v}]$ we have

$$\frac{dU^*(v)}{dv} = \frac{\bar{v} + \hat{v} - 2\underline{v}}{2(\bar{v} - \underline{v})}\alpha e^{-\alpha(v-B)}.$$

Since $U(\hat{v}, \hat{v}) = U^*(\hat{v})$ and $\hat{v} < \bar{v}$ then

$$\frac{1}{2}\alpha(\bar{v} + \hat{v} - 2\underline{v})e^{\alpha B} > e^{\alpha\hat{v}} - e^{\alpha\underline{v}}.$$

Multiplying both sides of this inequality by $e^{-\alpha\hat{v}}/(\bar{v}-\underline{v})$ yields

$$\begin{aligned}\left.\frac{dU^*(v)}{dv}\right|_{v=\hat{v}} &= \frac{\bar{v}+\hat{v}-2\underline{v}}{2(\bar{v}-\underline{v})}\alpha e^{-\alpha(\hat{v}-B)} \\ &> \frac{1}{\bar{v}-\underline{v}}(1-e^{-\alpha(\hat{v}-\underline{v})}) = \left.\frac{dU(v,v)}{dv}\right|_{v=\hat{v}}.\end{aligned}$$

Hence at $v=\hat{v}$ we have that $U^*(v)$ is steeper than $U(v,v)$. Furthermore, for $v<\hat{v}$, $U^*(v)$ is strictly concave in v and $U(v,v)$ is strictly convex in v. Hence for $v\in[B,\hat{v})$ we have $U^*(v)<U(v,v)$, which establishes (i).

We now prove (ii). Consider a bidder whose value is $v\in(\hat{v},\bar{v}]$. Given his rival follows t as given in Prop. 4(ii), the bidder's optimal threshold is either $t(\hat{v})$ or $\underline{v}$. For $v\in(\hat{v},\bar{v}]$ the bidder's payoff to a threshold of $t(\hat{v})$ is

$$\begin{aligned}U(\hat{v},v) &= \int_{\underline{v}}^{t(\hat{v})}(1-e^{-\alpha(v-x)})\frac{1}{\bar{v}-\underline{v}}dx + [F(\hat{v})-F(t(\hat{v})](1-e^{-\alpha(v-B)}) \\ &= \frac{1}{\bar{v}-\underline{v}}\left(e^{-\alpha v}\left[t(\hat{v})e^{\alpha B}-\frac{1}{\alpha}e^{\alpha t(\hat{v})}\right]+\hat{v}-\underline{v}+\frac{1}{\alpha}e^{-\alpha(v-\underline{v})}-\hat{v}e^{-\alpha(v-B)}\right).\end{aligned}$$

By (7)

$$t(\hat{v})e^{\alpha B}-\frac{1}{\alpha}e^{\alpha t(\hat{v})} = -\frac{1}{\alpha}e^{\alpha\hat{v}}+\hat{v}e^{\alpha B}.$$

Substituting and simplifying, we obtain

$$U(\hat{v},v) = \frac{1}{\bar{v}-\underline{v}}\left(\hat{v}-\underline{v}+\frac{1}{\alpha}e^{-\alpha(v-\underline{v})}-\frac{1}{\alpha}e^{-\alpha(v-\hat{v})}\right),$$

and hence

$$\frac{dU(\hat{v},v)}{dv} = \frac{1}{\bar{v}-\underline{v}}e^{-\alpha v}\left(e^{\alpha\hat{v}}-e^{\alpha\underline{v}}\right).$$

The bidder's payoff if he chooses threshold $\underline{v}$ is $U^*(v)$. We will establish that

$$\frac{dU(\hat{v},v)}{dv} = \frac{1}{\bar{v}-\underline{v}}e^{-\alpha v}\left(e^{\alpha\hat{v}}-e^{\alpha\underline{v}}\right) < \frac{\bar{v}+\hat{v}-2\underline{v}}{2(\bar{v}-\underline{v})}\alpha e^{-\alpha(v-B)} = \frac{dU^*(v)}{dv}. \tag{10}$$

Since $U(\hat{v},\hat{v})=U^*(\hat{v})$, this inequality implies that $U(\hat{v},v)<U^*(v)$ for $v\in(\hat{v},\bar{v}]$, which establishes (ii). We establish (10) by showing that

$$e^{\alpha\hat{v}}-e^{\alpha\underline{v}} < \frac{\bar{v}+\hat{v}-2\underline{v}}{2}\alpha e^{\alpha B}.$$

The equality $U(\hat{v},\hat{v}) = U^*(\hat{v})$ can be rewritten as

$$\hat{v} - \underline{v} + \frac{1}{\alpha}e^{-\alpha(\hat{v}-\underline{v})} - \frac{1}{\alpha} = \frac{\bar{v} + \hat{v} - 2\underline{v}}{2}(1 - e^{-\alpha(\hat{v}-B)}),$$

or

$$e^{\alpha\hat{v}} - e^{\alpha\underline{v}} = -\frac{\bar{v} + \hat{v} - 2\underline{v}}{2}\alpha e^{\alpha\hat{v}}(1 - e^{-\alpha(\hat{v}-B)}) + e^{\alpha\hat{v}}\alpha(\hat{v} - \underline{v}).$$

Hence to establish (10) we must show

$$-\frac{\bar{v} + \hat{v} - 2\underline{v}}{2}\alpha e^{\alpha\hat{v}}(1 - e^{-\alpha(\hat{v}-B)}) + e^{\alpha\hat{v}}\alpha(\hat{v} - \underline{v}) < \frac{\bar{v} + \hat{v} - 2\underline{v}}{2}\alpha e^{\alpha B}.$$

This inequality reduces to $\hat{v} < \bar{v}$, which clearly holds. □

Proof of Corollary 5: Case (i) Suppose $B \geq B_\alpha$. If a bidder's value is $v \leq B$ then his expected utility is

$$W(v) = \int_{\underline{v}}^{v} u(v - v_j)dF(v_j)$$

for both auctions. If $v > B$ then the bidder's expected utility in eBay is $W(v)$, since all types reject B. The bidder's expected utility in Yahoo is

$$Y(v) \equiv \int_{\underline{v}}^{t_\alpha(v)} u(v - v_j)dF(v_j) + \int_{t_\alpha(v)}^{v} u(v - B)dF(v_j),$$

where $t_\alpha(v)$ is given in Proposition 4(i). Hence

$$\begin{aligned} W(v) - Y(v) &= \int_{t_\alpha(v)}^{v} [-e^{-\alpha(v-v_j)} + e^{-\alpha(v-B)}]dF(v_j) \\ &= \frac{e^{-\alpha v}}{\bar{v} - \underline{v}}\left[(v - t_\alpha(v))e^{\alpha B} - \frac{1}{\alpha}e^{\alpha v} + \frac{1}{\alpha}e^{\alpha t_\alpha(v)}\right]. \end{aligned}$$

By equation (7), in the proof of Proposition 4, the term in square brackets is zero. This completes the proof for the case $B \geq B_\alpha$.

Case (ii): Suppose $B < B_\alpha$. Let c_α be as defined in Lemma 1. By Proposition 3 c_α is the symmetric equilibrium cutoff value in the eBay auction. By Proposition 4 c_α is the value at which the equilibrium threshold functions jumps down in the Yahoo auction. If $v \leq B$ then a bidder's expected utility is $W(v)$ for both auctions.

If $B < v < c_\alpha$ then the eBay expected utility equals the Yahoo expected utility $Y(v)$, just as in the no-jump case. If $v \geq c_\alpha$ then a bidder has expected utility

$$\frac{1}{2}\left(1 + F(c_\alpha)\right) u(v - B)$$

for both auctions. □

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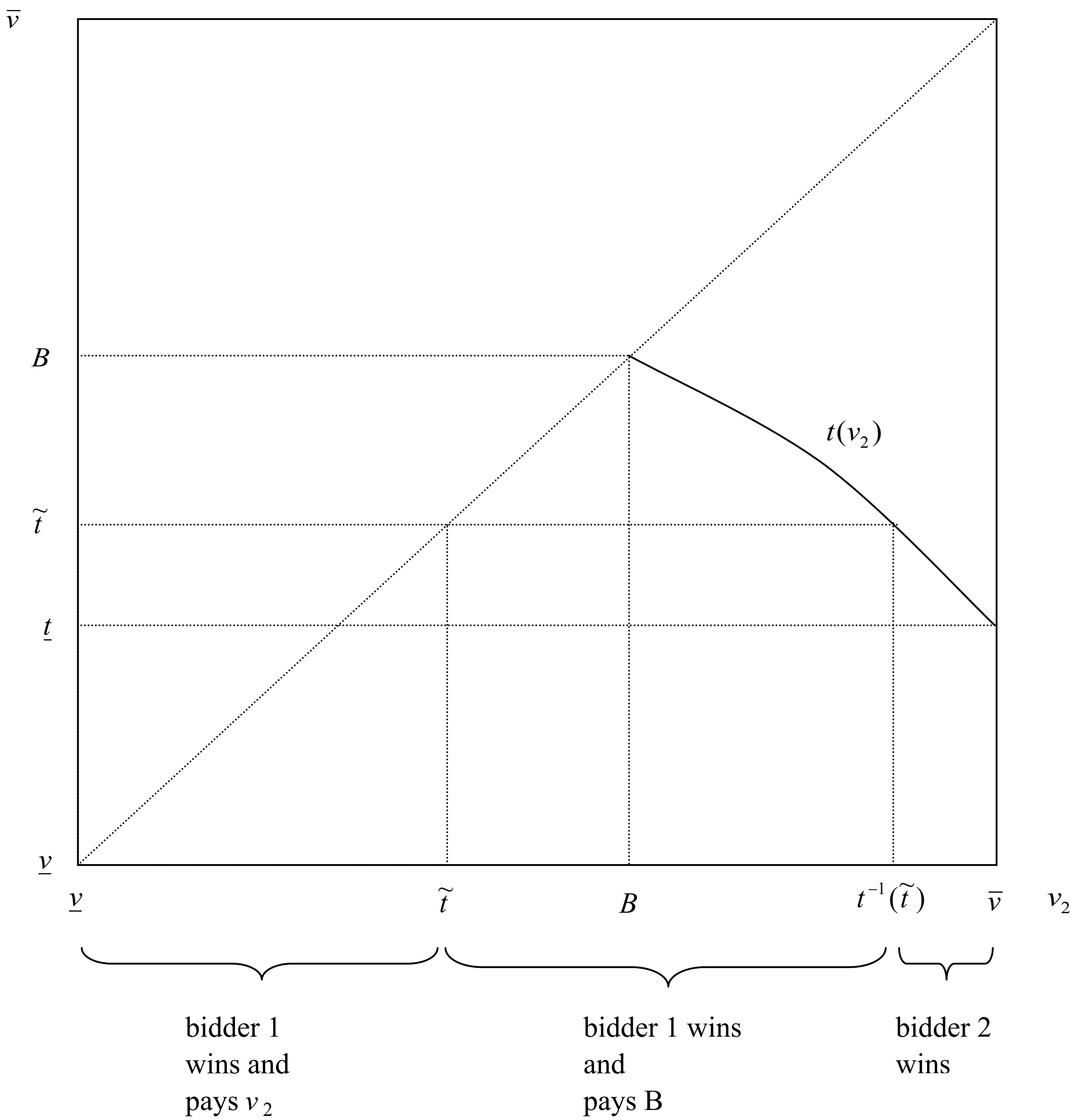
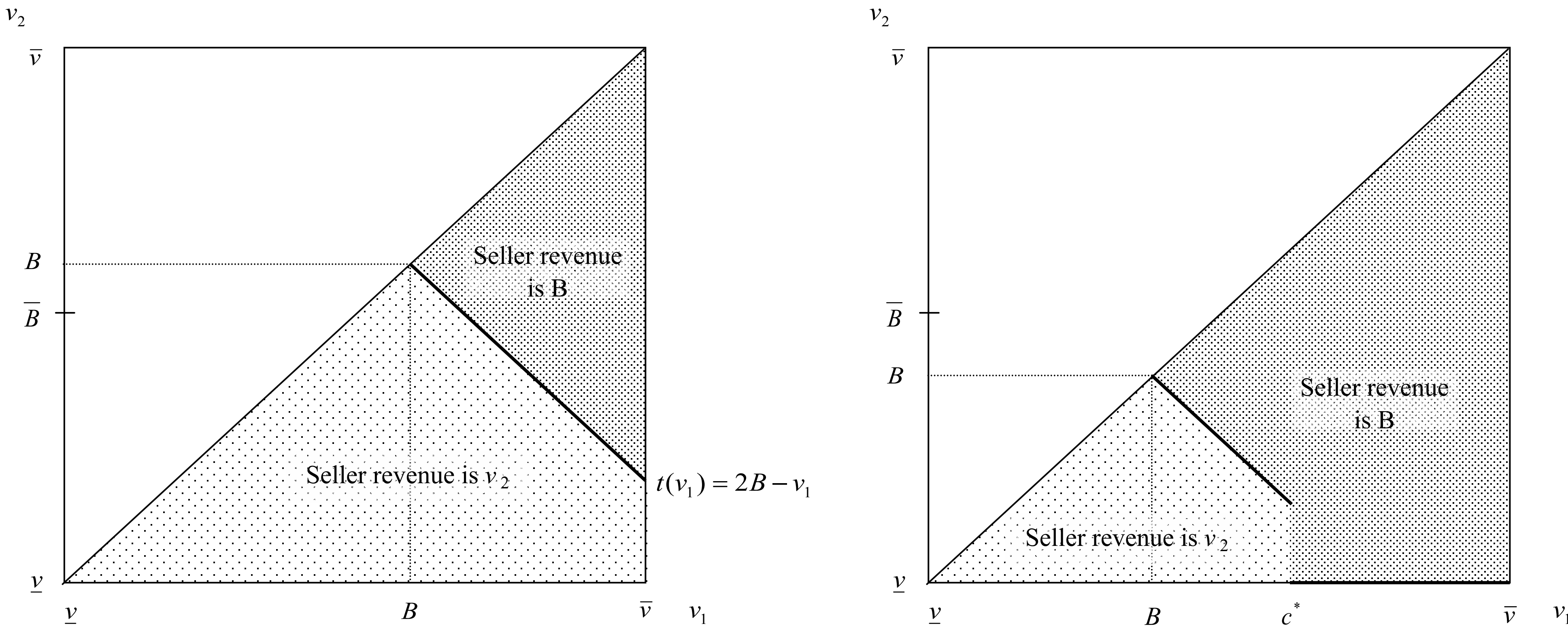

$\overline{v}$
B
$t(v_2)$
$\tilde{t}$
$\underline{t}$
$\underline{v}$
$\underline{v}$
$\tilde{t}$
B
$t^{-1}(\tilde{t})$
$\overline{v}$
v_2
bidder 1 wins and pays v_2
bidder 1 wins and pays B
bidder 2 wins

Figure 1



(a) $B > \overline{B}$

(b) $B < \overline{B}$

Figure 2: Seller Revenue in the Yahoo Buy-Now Auction

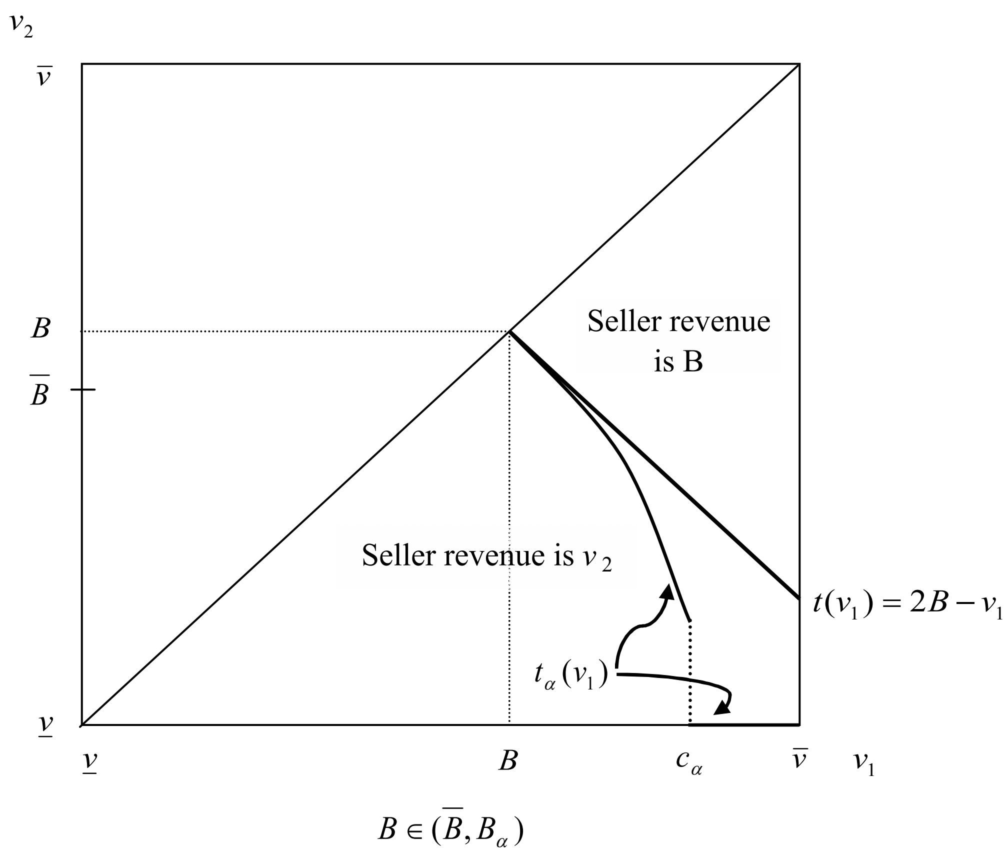


Figure 3: Comparing seller revenue in the Yahoo auction with risk averse and risk neutral bidders

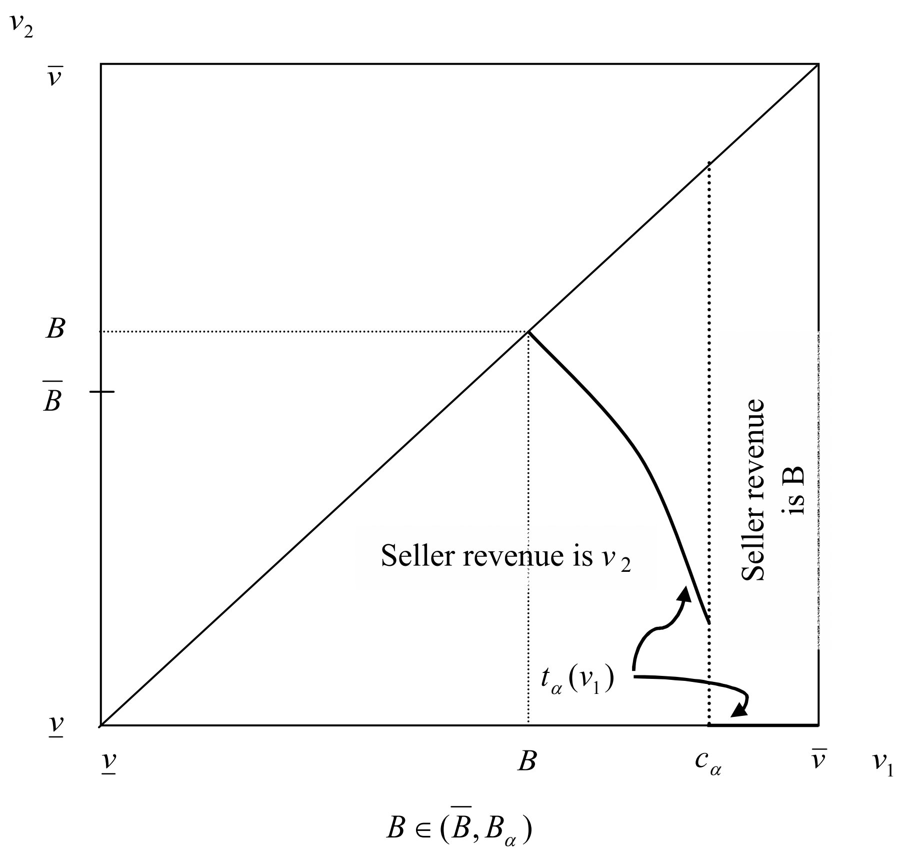
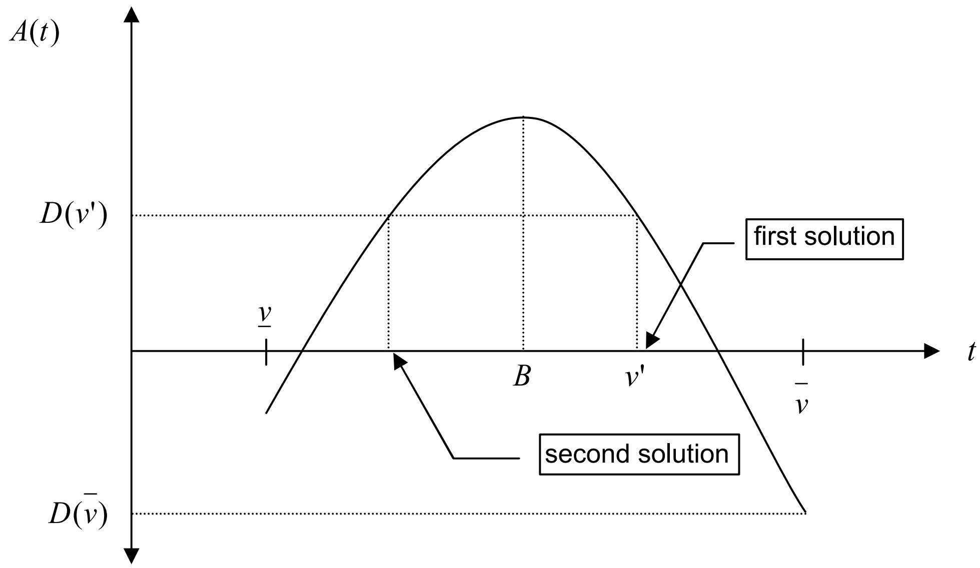


Figure 4: Comparing seller revenue in the Yahoo and eBay auctions when bidders are risk averse



$A(t)$
$D(v')$
$\underline{v}$
first solution
t
B
v'
$\bar{v}$
second solution
$D(\bar{v})$

Figure 5