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Asset Pricing with Heterogeneous Investment Horizons

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Abstract

We consider an analytically tractable asset pricing model describing the trading activity in a stylized market with two assets. Traders are boundedly rational expected utility maximizers with different beliefs about future prices and different investment horizons. In particular, we analyze the effects of the latter source of heterogeneity on the dynamics of price. We find that in the case with homogeneous agents, longer investment horizons lead to more stable dynamics. This is not true, however, in the case of a mixed population of traders, when the increase of heterogeneity in the investment horizons can introduce instability in the system. Furthermore, the role of heterogeneity turns out to be different for different trading behaviors and its effect on the aggregate dynamics depends on the whole ecology of agents' beliefs.

1 Introduction

In this paper we present a simple, analytically tractable, dynamic model of speculative market where mean-variance maximizing, boundedly rational agents trade in a Walrasian equilibrium framework. We use this endogenous asset pricing model to investigate the effect of heterogeneity in individual investment horizons on the aggregate market behavior.

As recognized by many authors, see, e.g., Brock (1997) and Guillaume et al. (1997), most of the "stylized facts" characterizing the dynamics of financial markets cannot be explained within the standard paradigm of the Efficient Market Hypothesis (see Fama (1970) for a review). This impossibility has been the driving force behind the emergence of a strand of "agent-based" models which describe markets as complex systems of interacting, boundedly rational and heterogeneous agents. The majority of contributions to the agent-based literature is constituted by relatively complex models which are mainly studied through extensive computer simulations, see for instance Levy et al. (1994), Arthur et al. (1997), Lux and Marchesi (1999), Kirman and Teyssière (2002) and the review in LeBaron (2006). In the same period a second group of similarly inspired models, the so-called "Heterogeneous Agent Models" (HAMs), emerged. These models typically present simplified frameworks which can be studied using the mathematical tools of the theory of dynamical systems. Examples of analytically tractable models can be found in the contributions by Lux (1995), Brock and Hommes (1998), Gaunersdorfer (2000), Chiarella and He (2001), Westerhoff and Reitz (2003), Brock et al. (2005), Anufriev et al. (2006), Chiarella et al. (2006), Manzan and Westerhoff (2007) and Anufriev and Bottazzi (2010).

The recent review by Hommes (2006) shows that the literature on HAMs has achieved a considerable success in explaining observed deviations from the Efficient Market Hypothesis. Apart from a few exceptions, however, HAMs focus on only one source of heterogeneity, the heterogeneity in expectations. Investors are typically assumed to be "myopic", so that their demand depends on the expectations about next period price. Such myopia implies homogeneity both in the time horizon and in the frequency with which investors operate in the market. This simplifying assumption is made despite the opposite evidence that, in real markets, "the variety of time horizons is large: from intra-day dealers, who close their positions every evening, to long-term investors and central banks" (Dacorogna et al., 1995). Furthermore, as early survey data on investors' expectations (Frankel and Froot, 1987b; Ito, 1990) and recent theoretical review (Kirman, 2006) suggest, heterogeneity in time horizons and heterogeneity in expectations are closely related, with the former

being plausibly one of the central components of the latter.

The question addressed in this paper is whether the predictions of the HAMs about the possible instability of the price dynamics due to the heterogeneity in expectations remain equally valid in the presence of traders with different investment horizons. Relatedly, our investigation sheds new light on the broader question of what are the minimal requirements sufficient to generate the "instability" phenomena observed in financial markets, like the emergence of bubble and burst dynamics and the persistence of excess volatility. While it is commonly accepted that the heterogeneity of behaviors characterizing the different traders plays a major role in the explanation of these "stylized facts", the question remains whether heterogeneity in traders' *expectations* is necessary to generate these phenomena, or, instead, other sources of heterogeneity (e.g., different attitude towards risk, different investment horizons, the extent and quality of available information) can be sufficient.

We analyze the effect of investors' horizons and their heterogeneity on the aggregate market dynamics within an asset-pricing model proposed in Bottazzi (2002). The structure of the model is simple. It is a speculative economy with two assets: a riskless bond and a risky equity whose price is determined via a Walrasian mechanism. The market participants maximize a mean-variance utility over their trading horizon but may be heterogeneous in terms of beliefs and preferences. Differently from a majority of HAMs, traders in the model of Bottazzi (2002) build expectations about future price return and not price levels. They also explicitly take into account both the expected return *and* the risk involved in their market positions. The relative shares of different trading behaviors are kept fixed.[1] Analysis in Bottazzi (2002) shows that the model with these characteristics can produce non-trivial aggregate price dynamics with sudden bubbles and crashes, when agents have heterogeneous expectations about return.

Within this framework we introduce two distinct classes of agents, *fundamentalists* and *chartists*, intended to stylize two basic attitudes of market participation. The first class represents traders who obtain future return predictions on the basis of the fundamental value of the asset, while traders belonging to the second class base their forecasts on the past history of price returns. The two groups of traders may differ in their investment horizons.

We show that if agents are homogeneous with respect to their pref-

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[1]The latter assumption can be motivated by the fact that the main players in the financial markets are usually led by specific strategies (often dependent on their horizon), whose frequency of updating is much lower than the frequency of trade. Brock and LeBaron (1996) discuss the issue of different time scales in finance in the context of asymmetric information model.

erences and the procedures by which they build their expectations, the heterogeneity in investment horizons cannot destabilize the market. This result tends to support the view that heterogeneity in expectations is of primary importance in generating excess volatility and trading volume. However, we also demonstrate that when agents do have heterogeneous expectations about future market behavior, the heterogeneity in time horizons has a strong effect on the aggregate behavior. In particular, the price dynamics turn out to be very sensitive to the way in which investors extrapolate the estimation of risk over time. We investigate this effect by studying two reasonable, albeit very simple, procedures for risk extrapolation and by comparing their respective effects on the dynamics of the model. Quite unexpectedly we find that for a relatively large range of parameters, the stability region shrinks with the increase of the investment horizon of fundamentalists. We argue that the ultimate reason for this phenomenon lies in the fact that the instability of the system is related with the fundamentalists' *relative demand* for the risky asset, with respect to the demand of chartists. If the former demand is relatively low, as it happens when the fundamentalists overestimate the risk, then the system becomes unstable.

The rest of the paper is organized as follows. In the next section we recall the importance of time horizon in investor's financial decision and briefly discuss previous contributions. In Section 3 the analytical asset pricing model is introduced. We discuss the different assumptions on which our model is based and, in particular, we describe two "stylized" classes of market participants, fundamentalists and chartists. Despite the simplicity of the model, we try to link the behavioral assumptions behind the definition of these two classes to empirical evidence. In this vein, we distinguish between two types of fundamentalists: sophisticated and unsophisticated. As a first simple example, in Section 4 we study the model in the special case in which there are no chartists in the market. We show that, in this case, heterogeneity in investment horizons cannot destabilize the market. In Section 5 we perform the study of the general case when both fundamentalists and chartists are present in the market. The analysis of the typical price and wealth dynamics generated by the model is presented in Section 6 while in Section 7 we discuss the implications of our findings with particular emphasis on the role of time horizons. Section 8 contains some final remarks and suggestions on possible future developments. The proofs of all propositions are provided in the Appendix.

2   Investment Horizons

Any financial advisor nowadays designs the portfolio of every individual client taking in consideration the horizon of his or her investment. The individual investment horizons, i.e., the span of time over which agents judge the performance of their investments, affect their perceived level of risk and, consequently, their portfolio choice. Moreover, different types of assets possess, in general, different level of desirability for traders who have long-term needs and for those who have short-term objectives.

Investors who have more than 10 years to invest in the market are usually thought of as those who have long horizon. These investors are typically young professionals or even high school or college students. There is quite strong evidence that financial planners encourage such young investors to have a portfolio with greater risk compared to other investors, since these higher risk portfolios tend also to generate higher market returns over time. In contrast, those investors who are of pre-retiree and retiree status usually have short investment horizons. Investors who have short horizons are generally the least tolerant toward investment risk. They are more concerned with preserving their existing capital and income. These two classes of investors judge risk differently, and so it is not surprising that the optimal choice for these two groups can be different.

Since the classical financial asset pricing model of Markowitz, Sharpe and Lintner, known as the Capital Asset Pricing Model (CAPM), did not capture the problem of different horizons in investment decisions, the natural question about its generalization arose in the academic community. Economic intuition would suggest that trading horizons do not affect prices in the world of fully rational traders, as even short-term speculators have to take into account future expected prices over an infinite time horizon. The intertemporal versions of the CAPM in LeRoy (1973) and Merton (1973), dynamic pricing models (Samuelson, 1969; Lucas, 1978) and more recent generalizations (see Campbell and Viceira (2002) and references therein) provide the conditions under which that intuition is correct. These models also offer a number of reasons why the optimal portfolio decision might be affected by investment horizons, including the time variations in the market expected returns and agent's labor income. However, all these models share extremely strict assumptions about the preference structure of agents and the price generation process. In particular, they share the assumption of an infinitely living representative agent having fully rational behavior.

Notwithstanding the simplicity and elegance of such an approach, the necessity to distinguish between different types of *boundedly rational* trading

behaviors is now widely acknowledged. Extensive evidence from real markets (Frankel and Froot, 1987a; Allen and Taylor, 1990) suggests that both technical and fundamental analysis play an important role for price determination. This evidence, indeed, was one of the driving forces behind the rise of the heterogeneous agents models mentioned in the previous section. Typically the HAMs ignore the fact that the heterogeneity of trading strategies can be related to differences in traders' investment horizons. This might be a drawback, because, e.g., simulations of an order-driven market in LiCalzi and Pellizzari (2003) show that the sole difference in planning horizons can lead to a rich set of different outcomes.

For what concerns analytical studies, a number of more traditional contributions tried to incorporate differences in trading horizons. For example, the model in Froot et al. (1992) considers an order-driven market of a single asset. It is based on the idea that short-term traders typically take advantage of the temporary rise of asset price generated by herding behavior, which, conversely, negatively affects the realized profits of long-term traders. In the model, before the trade starts, rational homogeneous speculators have to choose one among two alternative but complementary sources of information about the liquidation value of the asset. They then submit their optimal asset demands to the risk neutral competitive market-makers. It is shown that the equilibrium choices regarding the information source depend on traders horizons, i.e., on whether they are going to close their positions before or after the liquidation value becomes known. Short term traders prefer to coordinate their choices on one particular source of information. Such herding will lead to a temporary (positive or negative) "bubble" which turns out to be profitable to the majority of traders. On the other hand, traders with long horizons prefer efficient information, so that they tend to distribute themselves evenly over the two sources of information. By affecting traders expectations, investment horizons exert a direct effect on prevailing prices.

Osler (1995) makes a step further and considers the situation in which the aggregate market behavior endogenously affects the choice of traders' horizons. She presents a Walrasian two-period model in which two equilibria are possible, with a majority of long- or short-term traders, respectively. The key assumption for this result is the proportionality between the number of traders having a given horizon and the volatility of the market over that horizon. If the majority of traders switch to long horizons, the volatility of long-term returns decreases with respect to the short-term volatility, so that long-term traders will earn a higher risk-weighted expected return. On the other hand, when the majority of traders have short horizons, the lower short term volatility leads to an increase of the short-term traders expected gains. In this model

the aggregate price dynamics are determined simultaneously with the traders' horizons.

Hillebrand and Wenzelburger (2006) extend the traditional CAPM to the case of overlapping cohorts of investors with linear mean-variance preferences and different planning horizons. They present a rather general theoretical framework, but restrict their analysis to the case of "perfect forecasting rules" which is equivalent, in their framework, to rational expectations. In general, they find that short horizon investors prefer less risky portfolio and that the optimal portfolio for investors with different planning horizons is composed of different mixtures of the underlying risky assets.

In the present paper we try to bring the analysis of the effect of investment horizon on price dynamics within the framework of analytically tractable heterogeneous agent models. The dynamic aspect, coming from the presence of adaptive agents, distinguishes our model from the analysis of Froot et al. (1992), Osler (1995) and Hillebrand and Wenzelburger (2006), where the assumption about agents' rationality allowed to compute *static* market equilibria. We model the demand of agents through mean-variance optimization, allowing for heterogeneity in individual expectations about the first two moments of future returns. In particular, we introduce two classes of agents, fundamentalists and chartists. Even if the model with endogenous investment horizons of heterogeneous agents would be an interesting research topic, we shall fix in this paper the horizon for each type and concentrate on the effect of the horizons on the price dynamics. There are, however, many different ways in which the trading activity of agents with different investment horizons can be described.

One can assume, for instance, that an agent with a time horizon of $\eta > 0$ periods forward does not correct his portfolio between time t, when his decision about portfolio is initially made, and time $t + \eta$. This agent will participate in the market activity only once each η periods and will choose the desirable amount of risky asset maximizing the expected utility of his portfolio η periods in the future. This assumption introduces some level of "irrationality" in the agent's behavior, since the agent is supposed to ignore the information revealed by the trading activity that takes place in the $\eta - 1$ trading sessions occurring between his consecutive market participations. On the other hand, due to its relative simplicity and its low computational requirement, this behavior may not be unrealistic, especially for individual investors.

Another, rather extreme, possibility is to assume that an agent trades at each period, continuously correcting his portfolio composition in order to maximize the utility of future wealth, but taking into account the fact that the portfolio will be revised each period, on the basis of new information. Such an

approach assumes a high level of rationality in agents' description, and it has been widely used (in continuous-time case) after the seminal paper of Merton (1973). Even inside the representative agent paradigm, however, Merton's model is, in general, not analytically tractable. Moreover, the introduction of a minimal degree of heterogeneity in agents' behaviors would lead to even more complex dynamic programming problems that, in our opinion, would require a quite unrealistic degree of sophistication from the part of the agents.

In our model we shall follow a different, in a sense intermediate, approach. We assume that an agent with investment horizon η maximizes, at period t, his expected wealth at period $t + \eta$, without taking into account the possibility of future portfolio revisions. However, in each period the agent revises his portfolio if necessary, i.e., if the new market situation or new individual expectations lead him to a different optimal portfolio composition. This is similar to the strategies of the mutual funds which offer different investment plans for the specific time periods to individual investors and then operate in the financial markets on behalf of their clients.[2]

3 Model Structure

This section provides a detailed description of the formal implementation of the model. The different parameters are introduced and discussed and the general pricing equation for an arbitrarily large number of heterogeneous agents is obtained. Next, we introduce two special classes of traders, chartists and fundamentalists, and derive the reduced-form pricing equation which describes the dynamics of the model when only these types of agent are present in the market.

3.1 Market Dynamics

Consider a simple pure exchange economy with two goods: a riskless security (bond) that gives a constant interest rate $r_f > 0$ and a risky security (equity) that pays a positive dividend at the beginning of each period t. For the sake of simplicity we assume that the dividend is constant[3] and denote it as D.

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[2]Alternatively, our model can by formally presented as an overlapping generation model where, for any horizon η, there exist η generations of traders who revise their positions every η periods.

[3]The qualitative aspects of the model will be the same when the dividend is an i.i.d. random variable with moderately small variance (see Bottazzi (2002) for the details). In Section 6 we show the results of numerical simulations with random dividend and discuss the effect of the stochastic components on the aggregate dynamics.

This assumption is satisfied, for example, in the bond market where the fixed coupon payment plays the role of the dividend.

The riskless asset is assumed to be the numéraire of the economy and its price is fixed to 1. The price of the risky asset P_t is determined each period by market clearing. The *fundamental price* of the risky asset $\bar{P}$ is defined as the discounted stream of expected dividends

$$\bar{P} = \sum_{t=1}^{\infty} \frac{D}{(1 + r_f)^t} = \frac{D}{r_f} \,. \tag{1}$$

We also denote as $\rho_{t,t+\eta}$ the price return of the risky asset between time t and time $t + \eta$,

$$\rho_{t,t+\eta} = \frac{P_{t+\eta} - P_t}{P_t} = \prod_{\tau=t}^{t+\eta-1} (1 + \rho_{\tau,\tau+1}) - 1 \,.$$

Assume the economy is populated by N heterogeneous traders and let $W_{t,n}$ be the wealth of agent n $(1 \le n \le N)$ at time t. The investment decision of the n-th agent is described as the share $x_{t,n}$ of the personal current wealth invested in the risky asset. If η_n is the investment horizon, the decision will be based on the distribution of agent's wealth η_n periods in the future, $W_{t+\eta_n,n}$. Assuming that the earned dividends are fully reinvested in the riskless asset one has

$$W_{t+\eta_n,n}(P_t, \rho_{t,t+\eta}) =$$
$$(1 - x_{t,n}) W_{t,n} (1 + r_f)^{\eta_n} + x_{t,n} W_{t,n} \left(1 + \rho_{t,t+\eta_n} + \frac{D_{\eta_n}}{P_t} \right) , \tag{2}$$

where D_η stands for the stream of dividends earned in η periods, augmented by the received riskless interests

$$D_\eta = \sum_{\tau=0}^{\eta-1} D (1 + r_f)^\tau = \bar{P} \left((1 + r_f)^\eta - 1 \right) \,.$$

At each period t, and for any notional price P, the n-th agent will choose the share of wealth invested in the risky asset, $x_{t,n}^*$, to maximize the expected mean-variance utility of the future wealth (2), so that

$$x_{t,n}^*(P) = \underset{x_{t,n}}{\mathrm{argmax}} \left\{ \mathrm{E}_{t,n} \left[W_{t+\eta_n,n}(P, \rho_{t,t+\eta}) \right] - \right.$$
$$\left. - \frac{\beta_n}{2} \mathrm{V}_{t,n} \left[W_{t+\eta_n,n}(P, \rho_{t,t+\eta}) \right] \right\} , \tag{3}$$

where $E_{t,n}[\cdot]$ and $V_{t,n}[\cdot]$ stand, respectively, for the expected mean and variance of their argument obtained by agent n using the information available at time t, and β_n is the risk aversion parameter of agent n.

The intertemporal dynamics of the model proceeds as follows. At the beginning of period t agent n possesses $A_{t-1,n}$ shares of the risky and $B_{t-1,n}$ shares of the riskless asset. First, all agents receive dividends per every owned share of the risky asset and interest rate on owned riskless securities. These earnings are paid in terms of the numéraire. Then using (3) each agent computes the desired amount of the risky asset $A_{t,n}(P)$ for any notional price

$$
A_{t,n}(P) = \frac{x_{t,n}^*(P)\, W_{t,n}(P)}{P} \; . \tag{4}
$$

Finally, the prevailing price of the risky asset P_t is fixed using the market clearing condition

$$
\sum_{n=1}^{N} A_{t,n}(P_t) = A_t \; , \tag{5}
$$

where A_t is the number of outstanding shares of the risky security at time t. Alternatively, one can compute the individual *excess* demand function $\Delta A_{t,n}(P) = A_{t,n}(P) - A_{t-1,n}$ of agent n and fix the price of the asset using

$$
\sum_{n=1}^{N} \Delta A_{t,n}(P) = \Delta A_t \; , \tag{6}
$$

where $\Delta A_t = A_t - A_{t-1}$ is the "outside" supply of the risky shares at time t. Equations (5) and (6) are equivalent and establish, at the same time, the present price P_t of the risky asset and the present composition of each portfolio, agent n owning $A_{t,n} = A_{t,n}(P_t)$ risky and $B_{t,n} = P_t \Delta A_{t,n} + A_{t-1,n}\, D + B_{t-1,n}\, (1 + r_f)$ riskless shares. At this point, once prices and allocations are decided, *if the market behavior in terms of price and returns did not change*, agent n would avoid to trade in the market until period $t + \eta_n$. However, new information about realized prices may (and generally will) force the agent to change the desirable amount of the risky asset and, consequently, the composition of his portfolio, before the notional maturity of its present investment. Nonetheless, contrary to the dynamical programming approach, agents in our model do not take into account (and do not anticipate) their future actions. In this sense, our agents are boundedly rational.

3.2 Pricing Equation

In order to solve (3), agents have to form expectations about their wealth at future time. The only source of uncertainty is constituted by the return $\rho_{t,t+\eta_n}$. Let $\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}]$ and $\mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]$ be the expectation of agent n about the average value of price return and its variance, respectively. Then from (2), the conditional distribution of future wealth $W_{t+\eta_n,n}$, given current price P_t and wealth $W_{t,n}$, has the following expected value and variance

$$
\begin{aligned}
\mathrm{E}_{t,n}[W_{t+\eta_n,n}] &= W_{t,n}\,(1+r_f)^{\eta_n}+ \\
&\qquad + x_{t,n}\,W_{t,n}\left(\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}]+R_{\eta_n}(\bar{P}/P-1)\right) \quad (7)\\
\mathrm{V}_{t,n}[W_{t+\eta_n,n}] &= \left(x_{t,n}\,W_{t,n}\right)^2 \mathrm{V}_{t,n}[\rho_{t,t+\eta_n}],
\end{aligned}
$$

where $\bar{P}$ stands for the fundamental price of the risky asset introduced in (1), and $R_\eta = (1+r_f)^\eta - 1$ for the positive return gained from holding the riskless asset during η periods. Plugging (7) in (3), solving the optimization problem and substituting the result in (4), one obtains

$$
A_{t,n}(P) = \frac{\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}]+R_{\eta_n}(\bar{P}/P-1)}{\beta_n\,P\,\mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]}\ . \quad (8)
$$

Notice that if $P = \bar{P}$ and $\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}] = 0$, that is if the initial price of the risky security is equal to its fundamental value and no price change is predicted, and if the forecasted variance is positive $\mathrm{V}_{t,n} > 0$, then the desirable amount of the risky asset in portfolio is equal to zero. This is as expected, since in this situation the risky security pays the same return as the riskless one and is, for any level of risk, less desirable. More generally, the desirable amount of risky asset is proportional to the excess return and inversely proportional to the expected risk. It also decreases with the current price P and with agent's risk aversion coefficient β_n. Inside the previous expression the investor's horizon η_n appears in three distinct places: it affects the expected excess return and the expected variance, since they both have to be computed over η_n periods, and it contributes to the wealth accumulation through the dividend re-investment accounted for by the term R_{η_n}.

    Starting from the amount of asset desired by each agent n at any price level P the prevailing price is finally computed using (5) (or equivalently (6)). Assuming that the outstanding number of shares is constant $A_t = A_{TOT}$, $\forall t$ and substituting (8) in (5) one obtains

$$
\bar{A} = \frac{1}{N}\sum_{n=1}^{N}\frac{\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}]-R_{\eta_n}}{\beta_n\,\mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]}\ \frac{1}{P}+\frac{1}{N}\sum_{n=1}^{N}\frac{\bar{P}\,R_{\eta_n}}{\beta_n\,\mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]}\ \frac{1}{P^2}\ , \quad (9)
$$

where $\bar{A} = A_{TOT}/N$ stands for the number of risky shares per investor. The two coefficients on the right hand side of (9) are population averages of different quantities. The numerator of the first summand is the expected *excess return* of the capital gain of the risky asset, while the numerator of the second summand represents the capital which agent n obtains from the stream of dividends payed by the risky asset over η_n periods. Both quantities are weighted by the risk factor $1/(\beta_n \mathrm{V}_{t,n}[\rho_{t,t+\eta_n}])$, which accounts for the agents' personal evaluation of the risk implied in keeping the risky stock in the portfolio. Since $\bar{P} > 0$, the second term in the right hand side of (9) is positive. Together with the positiveness of $\bar{A}$, this implies the existence and uniqueness of the positive solution of the quadratic[4] equation (9). This solution defines the market price P_t in period t. The market price increases with the average risk-weighted expected capital gain and with the payed dividend and decreases with the relative supply of the asset and with the increase of the forecasted risk. In Fig. 1 the price determination is illustrated when the horizon $\eta = 1$ and the expectations are homogeneous, i.e., when $\beta_n = 1$, $R_{\eta_n} = r_f$, $\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}] = \mathrm{E}[\rho]$ and $\mathrm{V}_{t,n}[\rho_{t,t+\eta_n}] = \mathrm{V}[\rho]$ for all n and t. Three demand curves for different values of the expected return $\mathrm{E}[\rho]$ are reported. Notice that if $\mathrm{E}[\rho] = 0$, the market clearing price is below the fundamental level $\bar{P}$. This is because the agents expect positive fluctuations and require a risk premium for holding a positive amount of shares in the market equilibrium. If the agents expected zero fluctuations, the inverse demand function would become flat, resulting in the intersection with supply at $P = \bar{P} = 1$ and the risk premium would disappear.

The latter fact is an important property of (9) and can be demonstrated in general case. Assume that the price of the asset fluctuates randomly around a constant level. Then, if all agents use unbiased estimators to predict future price returns, the investment horizon has no effect on the forecast of expected returns and $\mathrm{E}_{t,n}[\rho_{t,t+\eta_n}] = 0$, for all n. With this assumption, Eq. (9) reduces to

$$\bar{A} = \frac{1}{N} \sum_{n=1}^{N} \frac{R_{\eta_n}}{\beta_n \mathrm{V}_{t,n}[\rho_{t,t+\eta_n}]} \frac{\bar{P} - P}{P^2} .$$

If the fluctuations around the constant price level get smaller and the agents use consistent and unbiased estimators for the variance, the denominators

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[4]The non-linear pricing equation (9) can be contrasted with the linear pricing equation obtained in Brock and Hommes (1998) starting from very similar assumptions. The difference resides in our assumption that agents form expectations about future return, not price, and in the special choice made in that paper to set $A_{TOT} = 0$.
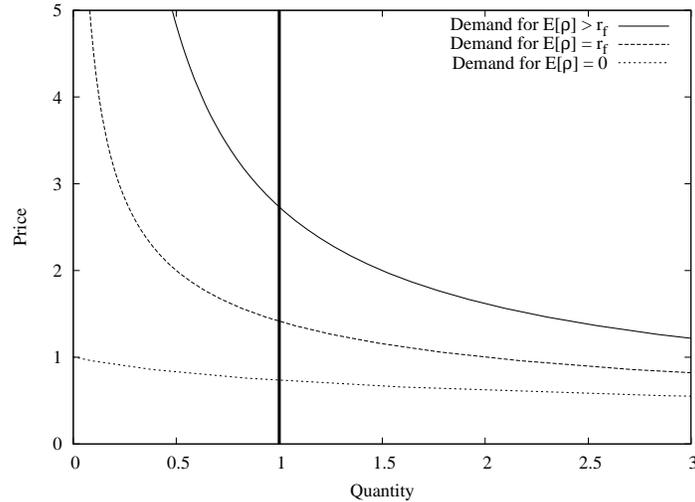
Figure 1: The inverse demand curve, A^{-1}, for the demand given from (9) as $A(P) = (\mathrm{E}[\rho] - r_f)/(\mathrm{V}[\rho]P) + r_f/(\mathrm{V}[\rho]P^2)$ is plotted for three different values of expected return $\mathrm{E}[\rho]$. The other parameters are: $\mathrm{V}[\rho] = 0.005$, $r_f = D = 0.01$, resulting in the fundamental price $\bar{P} = 1$. The vertical thick line corresponds to the constant supply of the risky asset.

of all the terms of the right hand side will converge to zero. Then (9) has a solution only if also the numerators in the terms of the right hand side converge to zero, i.e., $P \to \bar{P}$. Later, when the dynamical system will be fully specified, we will prove that, if adaptive traders are characterized by a minimal degree of procedural rationality (i.e., use reliable statistical estimators for future predictions) the system admits, as unique fixed point solution, the fundamental price of the asset. This discussion suggests that (9) is a reasonable behavioral extension of the "fundamental" evaluation of the asset provided in (1).

3.3   Types of Agents: Chartists and Fundamentalists

For analytical tractability we are forced to simplify the pricing equation (9). First of all, we assume that $\beta_n = \beta$ for all n. Second, we confine our attention to the case in which only two types of agents, *chartists* and *fundamentalists*, trade in the market. These types differ in their assumption about the underlying stochastic processes driving the dynamics of price. Furthermore, we will distinguish two types of fundamentalists according to the rule which they

adopt to extrapolate the prediction of the next period return and variance over longer time horizons. Below we specify, for each group, how expectations about the mean and variance of future returns are formed.

3.3.1 Chartists

Chartists predict returns on the basis of the past returns history using consistent statistical estimators. More precisely, at period t, they assume that the next price return is equal to the exponentially weighted moving average (EWMA) of past returns, based upon past price realizations up to P_{t-1}, and denoted with y_{t-1}, Analogously, for predicting the next period variance they compute the EWMA estimator of historical variances, denoted with z_{t-1}. The EWMA estimators y_{t-1} and z_{t-1} read

$$
\begin{aligned}
y_{t-1} &= (1-\lambda)\sum_{\tau=2}^{\infty} \lambda^{\tau-2}\rho_{t-\tau}\,, \\
z_{t-1} &= (1-\lambda)\sum_{\tau=2}^{\infty} \lambda^{\tau-2}[\rho_{t-\tau} - y_{t-1}]^2\,.
\end{aligned}
\tag{10}
$$

The parameter $\lambda \in [0,1)$ measures the relative importance agents put on different past observations: the last available return has the highest weight, while the weights of previous realized returns decline geometrically in the past. One can interpret λ as a "memory" parameter: the higher the value of λ is, the less the recent observations weigh. In the extreme case, when $\lambda = 0$, the agent's memory is the shortest possible and the last realized return is used as predictor of the next return. This is the case of "naïve" expectations. Notice that expressions above are analogous to the ones proposed by the RiskMetrics Group[TM] (J.P.Morgan, 1996), and widely applied by real operators in their forecasting activity.[5] In what follows we will use the recursive form of the previous two equations, namely

$$
\begin{aligned}
y_{t-1} &= \lambda y_{t-2} + (1-\lambda)\rho_{t-2}\,, \\
z_{t-1} &= \lambda z_{t-2} + \lambda(1-\lambda)(\rho_{t-2} - y_{t-2})^2\,.
\end{aligned}
\tag{11}
$$

Equations (11) describe a one period ahead forecast. To build a long-term forecast, chartists assume that future price dynamics can be described as a geometric random walk. Then the forecasts for both the expected return and

---

[5]The RiskMetrics Group[TM] group actually proposes an EWMA estimator of the volatility, defined as the second moment of the returns distribution. The expression above represents its natural extension to the central moment.

variance over η periods can be simply obtained multiplying the one-period forecast by a factor η, to read

$$\mathrm{E}_t^{\mathrm{C}}[\rho_{t,t+\eta}] = \eta \; \mathrm{E}_t^{\mathrm{C}}[\rho_{t,t+1}] = \eta \, y_{t-1} \tag{12}$$

and

$$\mathrm{V}_t^{\mathrm{C}}[\rho_{t,t+\eta}] = \eta \; \mathrm{V}_t^{\mathrm{C}}[\rho_{t,t+1}] = \eta \, z_{t-1} \,, \tag{13}$$

where symbols $\mathrm{E}_t^{\mathrm{C}}$ and $\mathrm{V}_t^{\mathrm{C}}$ denote the corresponding expectations of chartists. The individual demand function of a chartist can then be obtained from (8) and reads

$$A_t^{\mathrm{C}}(P) = \frac{1}{P} \frac{y_{t-1}}{\beta z_{t-1}} + \frac{\bar{P} - P}{P^2} \frac{B^{\mathrm{C}}(\eta)}{\beta z_{t-1}} \,, \tag{14}$$

where the term

$$B^{\mathrm{C}}(\eta) = \frac{(1 + r_f)^\eta - 1}{\eta} \tag{15}$$

accounts for the dependence of the demand function on the horizon η.

3.3.2   Fundamentalists

Fundamentalists believe that the price is governed by a process which constantly reverts to the fundamental value $\bar{P}$: if the asset is presently undervalued, its price will increase, and if it is overvalued, the price will fall. More precisely, they expect that the future price P_{t+1} will be, on average, between the current price P_t and the fundamental price $\bar{P}$

$$\mathrm{E}_t^{\mathrm{F}}[P_{t+1}] = P_t + \theta \, (\bar{P} - P_t) \,, \tag{16}$$

where $\theta \in [0, 1]$ describes the belief of fundamentalists about market reactivity in recovering the fundamental price.[6] When $\theta = 0$ equation (16) gives the so-called "naïve" expectations $\mathrm{E}_t^{\mathrm{F}}[P_{t+1}] = P_t$, while $\theta = 1$ corresponds to the case when fundamentalists believe that the fundamental value will be already reached in the next period. The expression of the multi-steps expected return under the assumption of mean-reverting market dynamics reads

$$\mathrm{E}_t^{\mathrm{F}}[\rho_{t,t+\eta}] = \left(1 - (1 - \theta)^\eta\right)\left(\frac{\bar{P}}{P_t} - 1\right). \tag{17}$$

---

[6]In contrast to the chartists' behavior, price P_t is included in the information set of fundamentalists. Such asymmetry simplifies computations and is not crucial for our results.

This result can be straightforwardly obtained by the recursive use of equation (16).

Since the volatility of the asset is determined essentially by the opinion of the market, we assume that fundamentalists forecast the one period ahead volatility in the same way as chartists. Thus, $V_t^F[\rho_{t,t+1}] = z_{t-1}$, where z_{t-1} is given in (10).

At this stage, it remains to specify the fundamentalists' forecast $V_t^F[\rho_{t,t+\eta}]$ for the variance of return over η periods. For this purpose we introduce two different types of fundamentalists. We refer to one type as *sophisticated* fundamentalists and to the other type as *unsophisticated* fundamentalists, according to the complexity of the analytic tools which they use to compute their forecast about future variance.

Unsophisticated Fundamentalists. These agents implement a relatively simple reasoning, assuming like chartists that the variance of return linearly increases with time

$$V_t^{UF}[\rho_{t,t+\eta}] = \eta \ V_t^F[\rho_{t,t+1}] = \eta \ z_{t-1} \ . \tag{18}$$

On one hand, as we will show below, this assumption about fundamentalists' forecast contradicts to the expectations about price behavior in (16). On the other hand, this specification of the variance corresponds to the behavior commonly found among financial investors who tend to use fundamental evaluation in judging different investment opportunities while preferring an econometric (technical) approach for the evaluation of the associated risk. Moreover, the Brownian scaling of the volatility described in (18) is qualitatively similar to the one actually found for real markets (Dacorogna et al., 2001).

The individual demand function of an unsophisticated fundamentalist can then be obtained from (8) and reads

$$A_t^{UF}(P) = \frac{\bar{P} - P}{P^2} \frac{B^{UF}(\eta, \theta)}{\beta z_{t-1}} \ , \tag{19}$$

where the term

$$B^{UF}(\eta, \theta) = \frac{(1 + r_f)^\eta - (1 - \theta)^\eta}{\eta} \tag{20}$$

accounts for the dependence of the demand function on the horizon η.

Sophisticated Fundamentalists. These fundamentalists forecast future returns variance in a way which is consistent with their return expectations (17). They believe that (16) describes the price dynamics *in each period*. This assumption allows them to model the forecast of time series of future returns as a mean reverting stochastic process. In Appendix A we derive the Fokker-Planck equation that describes the evolution of the long-horizon forecast and we show that, first, the prediction for the long-term return satisfies (17), and, second, that the long-term asset volatility forecast can be obtained from the one-period volatility forecast $V_t^{\mathrm{F}}[\rho_{t,t+1}] = z_{t-1}$ according to the formula

$$V_t^{\mathrm{SF}}[\rho_{t,t+\eta}] = \frac{1 - (1-\theta)^{2\eta}}{\theta(2-\theta)} z_{t-1} . \tag{21}$$

The crucial difference between (18) and (21) consists in the fact that the former infinitely increases with fundamentalists' investment horizon η, while the latter converges to the asymptotic value $z_{t-1}/(\theta(2-\theta))$. Thus, on longer investment horizons, *unsophisticated fundamentalists overestimate risk*. We will see later that this difference leads to crucial change in the dynamics of the model.

Finally, the individual demand function of sophisticated fundamentalists can be obtained from (8) and reads

$$A_t^{\mathrm{SF}}(P) = \frac{\bar{P} - P}{P^2} \frac{B^{\mathrm{SF}}(\eta,\theta)}{\beta z_{t-1}} , \tag{22}$$

where the term

$$B^{\mathrm{SF}}(\eta,\theta) = \left((1+r_f)^\eta - (1-\theta)^\eta\right) \frac{1 - (1-\theta)^2}{1 - (1-\theta)^{2\eta}} \tag{23}$$

accounts for the dependence of the demand function on the horizon η.

The specification of sophisticated fundamentalists behavior concludes the building of our artificial market model. In what follows we analyze the effects of the investment horizons on the aggregate dynamics. For this purpose we present different models based on the previous assumptions, characterized by an increasing degree of complexity: in the next section we analyze the case of market without chartists and later, in Section 5, we will extend the analysis to the situation in which both chartists and fundamentalists participate the market.

4 Market without Chartists

This section is devoted to the simplified model in which the market is populated only by fundamentalists. We explicitly provide the analysis for the case when all fundamentalists are unsophisticated. One can easily check that our result does not change in the case of sophisticated fundamentalists.

4.1 Homogeneous Investment Horizons

Let us start with the simplest situation and assume that all unsophisticated fundamentalists in the market have the same forecasting parameter θ and the same horizon η. Substituting the demand function (19) in (9) the price of the risky asset P_t is determined as the positive root of

$$P^2 \beta \bar{A} z_{t-1} + (P - \bar{P}) B^{\text{UF}}(\eta, \theta) = 0 \, , \tag{24}$$

which reads

$$P_t = \frac{-r + \sqrt{r^2 + 4\, r\, \bar{P}\, \gamma\, z_{t-1}}}{2\, \gamma\, z_{t-1}} \, , \tag{25}$$

where the following parameters have been introduced

$$
\begin{aligned}
\gamma &= \beta \bar{A} \, , \\
r &= B^{\text{UF}}(\eta, \theta) \, .
\end{aligned} \tag{26}
$$

Using the pricing equation (25) and the recursive definition of estimators (11) the 3-dimensional system describing the dynamics of the market in terms of the scaled price $p_t = \gamma\, P_t$ reads

$$
\begin{cases}
p_{t+1} = f(z_t) = \dfrac{-r + \sqrt{r^2 + 4sz_t}}{2\, z_t} \\[2mm]
y_{t+1} = \lambda y_t + (1 - \lambda)\left(\dfrac{f(z_t)}{p_t} - 1\right) \\[2mm]
z_{t+1} = \lambda z_t + \lambda(1 - \lambda)\left(\dfrac{f(z_t)}{p_t} - 1 - y_t\right)^2 \, ,
\end{cases} \tag{27}
$$

where $s = r\gamma\bar{P}$. One has the following

Proposition 4.1. *The system in* (27) *has only one fixed point* $(\gamma\bar{P}, 0, 0)$ *corresponding to the fundamental price of the risky asset. This fixed point is locally asymptotically stable for all* $\lambda < 1$.

Proof. See Appendix B.   □

We run an extensive set of simulations with different parameter values and different initial conditions and we conjecture that the point $(\gamma \bar{P}, 0, 0)$ is, probably, even globally stable.

Notice that a variation of η leads to changes in the value of the parameter r, and hence in the value of s, but does not modify the system itself. Thus, the price converges to the fundamental value independently of the investment horizon. The speed of convergence to the fixed point depends, however, on η.

4.2   Heterogeneous Investment Horizons

Consider now the market composed of K groups of traders. Each group $i \in \{1, \ldots, K\}$ is composed by N_i fundamentalists with a forecasting parameter θ_i and an investment horizon equal to η_i. If we denote the population shares of the different groups as $f_i = N_i/N$, where $N = \sum_{i=1}^{K} N_i$, the market clearing condition becomes

$$P^2 \beta \bar{A} z_{t-1} + (P - \bar{P}) \sum_{i=1}^{K} f_i B^{\text{UF}}(\eta_i, \theta_i) = 0 . \tag{28}$$

There is an obvious similarity between (24) and (28). The dynamics of the system are still described by (27) with the only difference that the parameter r now reads

$$r = \sum_{i=1}^{K} f_i B^{\text{UF}}(\eta_i, \theta_i) . \tag{29}$$

Then, in the case of many groups of fundamentalists with different investment horizons, the asymptotic behavior of the system remains the same and the price converges to the fundamental value. In other words, in the market with fundamentalists, the heterogeneity in the investment horizons does not affect the stability of the system.

In this section we have shown that the presence of groups of fundamentalists with different time horizons changes the values of the parameters of the dynamical system but does not affect its qualitative behavior. Thus, we can conclude that there are no qualitative differences between the case with homogeneous (with respect to the investment horizons) fundamentalists and the case with heterogeneous ones. In both cases the price converges to the fundamental value and the expectation about future variance goes to zero.

This result reminds a classical result in financial economics, obtained using a completely different approach. The solution of the dynamic programming model with homogeneous expectations and under a specific assumption about preferences (power utility function) in Merton (1973) shows that the optimal choice of the investor, in each period, does not depend on his investment horizon. This implies that the dynamics of the model remains the same when populated by agents with different horizons. We reach the same qualitative conclusions, using a different utility function and in the presence of heterogeneous, boundedly rational agents.

5 Complete Model: Fundamentalists vs. Chartists

Below we derive the deterministic system describing the dynamics of market where both fundamentalists and chartists are present. We start with the case when all fundamentalists are non-sophisticated and then move to the case with sophisticated fundamentalists.

5.1 Unsophisticated Fundamentalists

Let us consider the market composed of N_1 unsophisticated fundamentalists with horizons η_1 and homogeneous forecasting parameter θ, characterized by individual demand (19), and N_2 chartists with horizons η_2 and whose demand function is given by (14). Denote as f_1 and f_2 the fractions of fundamentalists and chartists, respectively, and assume that all investors in the market have the same parameter λ. The market clearing equation reads

$$
\begin{aligned}
P^2 \beta \bar{A} z_{t-1} + P \left(f_1 B^{\mathrm{UF}}(\eta_1, \theta) + f_2 B^{\mathrm{C}}(\eta_2) - f_2 y_{t-1} \right) - \\
- \bar{P} \left(f_1 B^{\mathrm{UF}}(\eta_1, \theta) + f_2 B^{\mathrm{C}}(\eta_2) \right) = 0 \,.
\end{aligned}
\tag{30}
$$

The positive root of this equation, together with (11), completely define the dynamics of the system. Introducing $\gamma = \beta \bar{A} / f_2$ and

$$
r = \frac{f_1}{f_2} B^{\mathrm{UF}}(\eta_1, \theta) + B^{\mathrm{C}}(\eta_2) \,,
\tag{31}
$$

the 3-dimensional system for the evolution of the rescaled price $p(t) = \gamma P_t$ reads

$$
\begin{cases}
p_{t+1} = g(y_t, z_t) = \left(y_t - r + \sqrt{(y_t - r)^2 + 4sz_t} \right) / 2z_t \\
y_{t+1} = \lambda y_t + (1 - \lambda)\left(\dfrac{g(y_t, z_t)}{p_t} - 1 \right) \\
z_{t+1} = \lambda z_t + \lambda(1 - \lambda)\left(\dfrac{g(y_t, z_t)}{p_t} - 1 - y_t \right)^2 ,
\end{cases}
\tag{32}
$$

where, as in (27), $s = r\gamma\bar{P}$. Notice that (32) coincides with the system in Bottazzi (2002) derived under the assumption of myopic agents. One has the following

Proposition 5.1. *The system in* (32) *possesses an unique fixed point* $(\gamma\bar{P}, 0, 0)$. *This point is locally asymptotically stable if*

$$
r + \lambda > 1 .
\tag{33}
$$

When the inequality changes its sign, the complex eigenvalues of the Jacobian cross the unit circle, so that the system exhibits a Neimark-Sacker *bifurcation and locally stable quasi-periodic cycles emerge.*

Proof. See Appendix C in Bottazzi (2002). □

Notice that the presence of chartists does not change the fixed point solution of the system, so that at equilibrium the price of the risky asset stands again on the fundamental level. In general, when the value of λ and/or r decreases, the system loses stability. Thus, a reduction of the effective "memory" of chartists tends to destabilize the system. At the same time, since $\partial_{r_f} B^{\mathrm{UF}}, \partial_{r_f} B^{\mathrm{C}} > 0$ and r in (31) is increasing with B^{UF} and B^{C}, an increase of the riskless interest rate tends to stabilize it. The effect of the horizon parameters will be analyzed in Section 7.

5.2   Sophisticated Fundamentalists

It is easy to check that when fundamentalists in the market are sophisticated, i.e., when they use (21) instead of (18) to predict future variance, the pricing equation is very similar to the one obtained in the previous case. The only difference is that the coefficient B^{UF} in replaced by B^{SF}. The dynamical system describing the evolution of price is still given by (32), but the parameter r now reads

$$
r = \frac{f_1}{f_2} B^{\mathrm{SF}}(\eta_1, \theta) + B^{\mathrm{C}}(\eta_2) .
\tag{34}
$$

5.3   Generalization to Many Groups

The generalization to many groups of fundamentalist traders is straightforward. Let us consider a market populated by K_1 groups of unsophisticated fundamentalists, K_2 groups of sophisticated fundamentalists and a group of chartists. We denote with f_c the population share of chartist traders and with $\tilde{f}_i$ with $i \in \{1, \dots, K_1 + K_2\}$ the relative shares of the different groups of fundamentalists (i.e., the number of fundamentalists of group i over the number of chartists). Let η_c, θ_i and η_i for $i \in \{1, \dots, K_1 + K_2\}$ denote the behavioral parameters of chartists and of the different groups of fundamentalists, respectively. Then, the dynamics of the system is still described by (32) with $\gamma = \beta \bar{A}/f_c$ and

$$r = \sum_{i=1}^{K_1} \tilde{f}_i \, B^{\mathrm{UF}}(\eta_i, \theta_i) + \sum_{i=K_1+1}^{K_1+K_2} \tilde{f}_i \, B^{\mathrm{SF}}(\eta_i, \theta_i) + B^{\mathrm{C}}(\eta_c)\,. \tag{35}$$

Summarizing, we have shown that irrespectively of the number and type of fundamentalists, when chartists are present, the market dynamics is described by (32), and Proposition 5.1 applies.

6   Price and Wealth Dynamics

As shown in the previous section, the fundamental price of the risky asset remains a fixed point of the system irrespectively of the kind of agents considered, chartists, unsophisticated or sophisticated fundamentalists. Conversely the stability conditions of this fixed point depend on the different trading behaviors. The qualitative behavior of system (32) is studied at length in Bottazzi (2002). In general, depending on parameter values and initial conditions, it can display convergence to a fixed point, quasi-cyclic or chaotic behavior.

As an example, in Fig. 2 we plot the long run price dynamics obtained with different sets of parameters (see caption). In the benchmark case (when all agents have investment horizons equal to one, i.e., $\eta_1 = \eta_2 = 1$) price convergences to the fundamental value. The increase of the investment horizon of fundamentalists destabilizes the fundamental equilibrium. This plot illustrates also one of the typical non-converging patterns of price generated by system (32) – price follows quasi-periodic behavior: after relatively slow rise price suddenly falls. This behavior reminds the crashes after "speculative bubbles" observed in real financial markets.
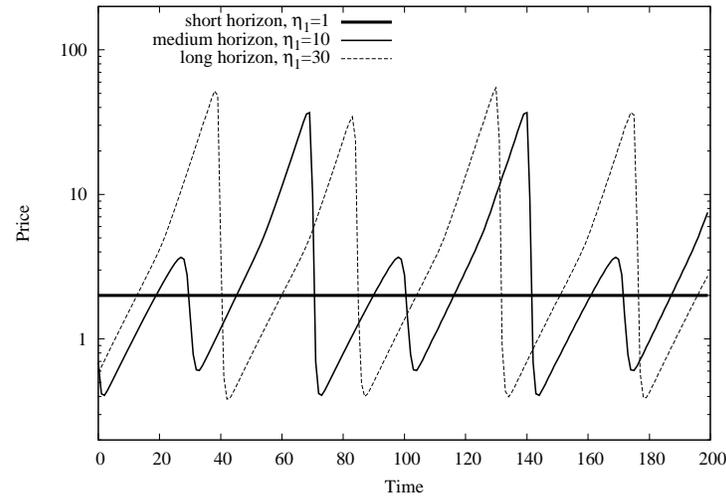
Figure 2: Price dynamics generated by system (32) (after a transitory period of 1000 steps) for different values of fundamentalists' investment horizon. With $\eta_1 = 1$ the price converges to the fundamental value, but with $\eta_1 = 10$ or $\eta_1 = 30$ the price fluctuates around that value with periodic crashes and booms. Parameters are $r_f = 0.05$, $D = 0.1$, $\bar{A} = 2$, $\beta = 1$, $\lambda = 0.85$, $f_1 = 0.2$, $\theta = 0.4$, $\eta_2 = 1$. Initial conditions are $P = 1$, $y = 0.01$, $z = 0.0001$.

In Fig. 3 we report the price history generated by (32) where the constant value of dividend has been replaced with a random variable D_t independently extracted at each time step from a log-normal distribution. A random variable D_t has mean $D = 0.1$ and standard deviation $\sigma_D = 0.02$. As can be seen, the introduction of noise changes the shape and the details of the trajectories. However, even considering relatively large (2% in terms of the average dividend level) random shocks, the peculiar dynamics of booms and crashes are still clearly visible.[7] Notice that since (32) depends on dividend through the variable $s = r\beta\bar{A}D/(f_2 r_f)$, other sources of randomness, e.g., in the total supply $\bar{A}$, would lead to a similar effect.

In order to better understand the mechanisms leading to the observed aggregate dynamics, it is useful to analyze the individual behavior of the dif-

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[7]Our simulations show that when standard deviation of dividend is close to 0, the amplitude of oscillations for parameters corresponding to unstable specifications (e.g., $\eta_1 = 10$ or 30 in Fig. 3) is close to the deterministic case. When the standard deviation increases, the amplitude of oscillations decreases, as it is seen in Fig. 3. Ultimately, the exogenous noise introduced through the dividends process completely dominates the endogenous fluctuations of the model.
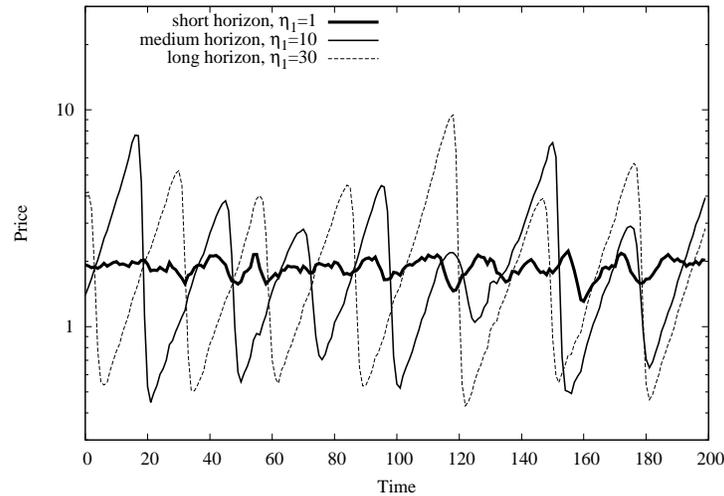
Figure 3: Price dynamics generated by a stochastic version of the system (32) (after a transitory period of 1000 steps) for different values of fundamentalists' investment horizon. Dividend realizations D_t are independently drawn from a log-normal distribution. The parameters of this distribution are chosen in such a way that the mean of dividend is 0.1 and standard deviation is 0.02. The values of the other parameters are the same as in Fig. 2.

ferent groups of agents. Consider for instance the amount of risky shares A_t owned by fundamentalists at time t. This is reported in Fig. 4 together with the price levels. The total amount of shares per trader is $\bar{A} = 2$. Negative shares correspond to short positions. Notice that in the period of increasing prices (for instance, steps $10 - 30$), the amount of shares in fundamentalists' portfolio is initially decreasing. Since chartists are willing to buy the risky asset, fundamentalists eventually reach short positions. On the contrary, the bursting of the speculative bubble corresponds to a reallocation of the risky asset toward fundamentalists, who now are willing to take long positions in an asset they consider undervalued.

It is interesting to know if and to what extent the joint dynamics of prices and portfolio composition is more valuable for one group of agents compared to the other. To investigate this issue consider the excess profit $\Pi_{t,n}$ received by agent n at time t

$$\Pi_{t,n} = W_{t+1,n} - (1 + r_f) W_{t,n} = A_{t,n} \left(P_{t+1} + D - P_t(1 + r_f) \right) . \qquad (36)$$

This quantity depends on both the dividend payment and the relative appreciation (or depreciation) of the risky asset. Notice that at equilibrium $P_t = \bar{P}$,
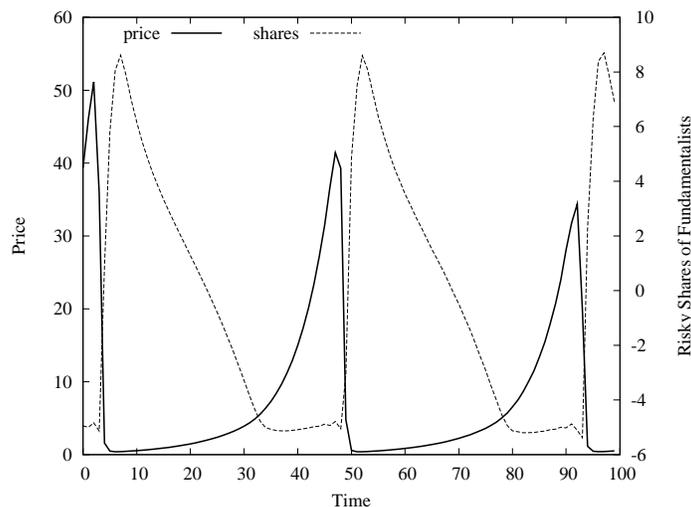
Figure 4: Price and number of shares of the risky asset in portfolio of fundamentalists (after a transitory period of 1000 steps). Parameters are set as in Fig. 2 with $\eta_1 = 30$.

the excess profit is zero for every trader. The dynamics of the excess profit for fundamentalists and chartists are reported in Fig. 5 with the same setting as in Fig. 4. As can be seen, fluctuations in excess profit are large. During the bubble, chartists enjoy better performances while fundamentalists constantly loose their wealth. The situation is abruptly reversed when the bubble bursts: then, in a few time steps, chartists incur huge losses and fundamentalists rapidly recover their previous losses. With similar dynamics repeating over time, no group can be said to outperform the others. The picture however changes if one considers the net earnings $E_{t,n}$ that the trading activity generates for the different agents. Net earnings are defined as the excess return over the amount of the riskless security owned at time t

$$E_{t,n} = B_{t+1,n} - (1 + r_f) B_{t,n} = A_{t,n} (D + P_{t+1}) - P_{t+1} A_{t+1,n} . \tag{37}$$

This quantity measures the flow of the riskless security which is generated by the trading activity in the market. Since, roughly speaking, one can think to the riskless security as the consumption good, this can be considered as an approximate measure of relative welfare. Notice that the total amount of earnings generated by the trading activity is always proportional to the total amount of paid dividend, $\sum_n E_{t,n} = A_{TOT} D$, irrespectively of the realized prices. The dynamics of earnings for chartists agents is reported in Fig. 6
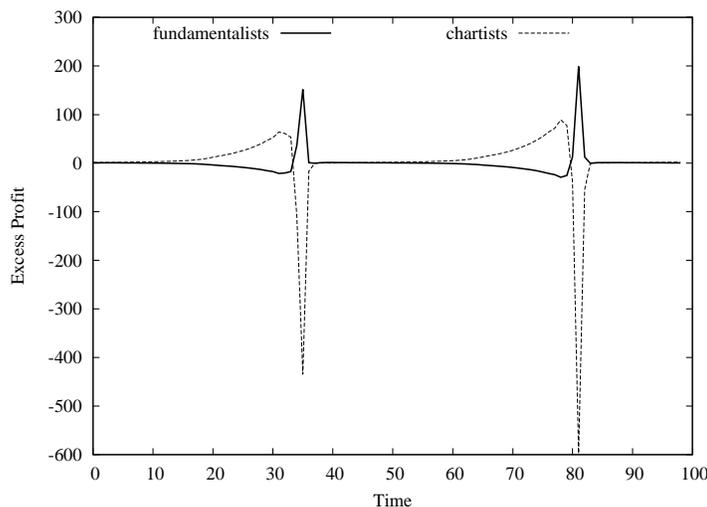
Figure 5: Dynamics of excess profit Π for the two groups of traders (after a transitory period of 1000 steps). Parameters are set as in Fig. 4.

together with the price of the asset. As can be seen, when the prices are low, chartists have relatively low earnings. Their earnings however increase in the second part of the bubble period where they display a more aggressive buy attitude due to low risk estimation. The positive earnings of chartists observed during the short crash period are generated by their fast discharge of risky assets.

7 Stability of fundamental price and investment horizons

In this section we will focus on the unique "fundamental" fixed point and on the effects that the investment horizons of traders have on its stability.

The inspection of condition (33) and the analysis of the different dependencies of parameter r defined in (35) reveal some general features of the model. First, the stability region of the fundamental fixed point enlarges when the parameter λ increases. This parameter describes the "smoothness" of agents' forecast and, roughly speaking, the length of their memory. Second, since the coefficients B^{C}, B^{UF} and B^{SF} defined respectively in (15), (20) and (23), are increasing functions of both r_f and θ, the stability region is larger when the values of these parameters are higher. As expected, both an increase
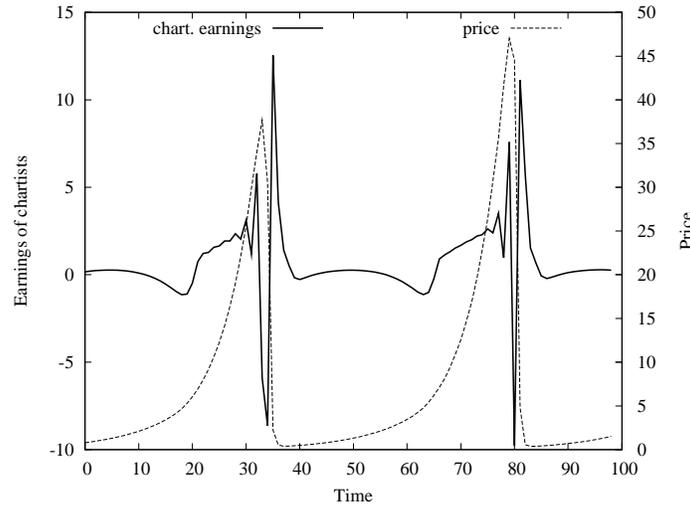
Figure 6: Dynamics of trading earnings E for the group of chartists compared with asset prices (after a transitory period of 1000 steps). Parameters are set as in Fig. 4.

in the riskless return and in the perceived efficiency of the market in restoring the fundamental price tend to stabilize the dynamics. Third, since

$$B^{\mathrm{C}}(\eta) \leq B^{\mathrm{UF}}(\eta, \theta) \qquad \text{and} \qquad B^{\mathrm{C}}(\eta) \leq B^{\mathrm{SF}}(\eta, \theta) \qquad \forall \eta, \, \forall \theta \ ,$$

an increase of the share of fundamentalists in the market will enlarge, *ceteris paribus*, the stability domain of the fixed point. These results are quite intuitive and do not require further comments. Their emergence can be traced back to the assumptions about agents' behavior and market structure.

Let us now turn to the analysis of the role of investment horizons. Notice that the only effect of the investment horizons on the stability of the system is through the definition of the parameter r which appears in (33). More precisely, the investment horizons of the different group of agents contribute to the definition of r via the coefficients B^{C}, B^{UF} and B^{SF} defined in Section 3.3.

First of all, we observe that when a single type of agents is present alone in market, would it be fundamentalist or chartist, an increase of the time horizon does not disturb the stability of the system. As already discussed in Section 4, when only fundamentalists are present in the market, the fixed point is always stable. Concerning the chartists, if one considers (31) and sets $f_1 = 0$, the parameter r only depends on term $B^{\mathrm{C}}(\eta_2)$. Since

$$\frac{dB^{\mathrm{C}}(\eta)}{d\eta} = \frac{(1 + r_f)^{\eta} \left(\eta \ln(1 + r_f) - 1 \right) + 1}{\eta} > 0 \qquad \forall \eta \geq 1 \ .$$
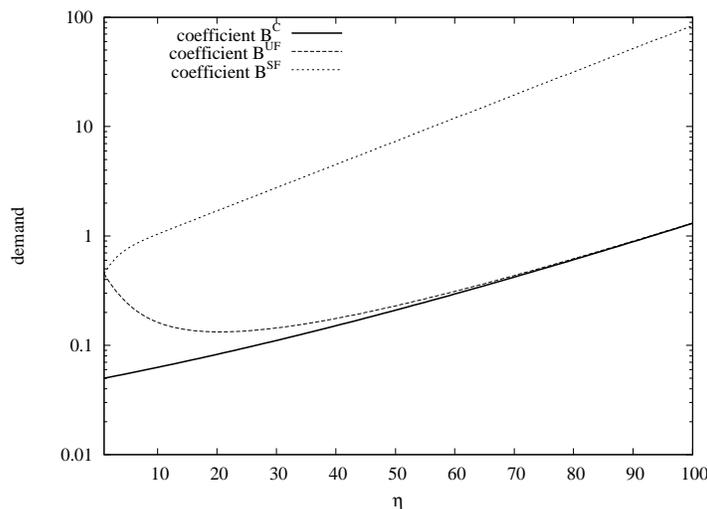
Figure 7: Behavior of coefficients B^{C}, B^{UF} and B^{SF} as functions of η for $\theta = 0.4$ and $r_f = 0.05$.

this term is increasing function of η_2. Thus, also in the case of chartists, an increase of the investment horizon enlarges the stability domain of the fixed point.

The last result can be immediately extended to the case in which both types are present in the market. Indeed the parameter r in (35) is an increasing function of η_c. Consequently, an increase of chartists' horizon always stabilizes the market, see illustration in Fig. 7, where the η-dependent factors of the different demand functions are shown.

The same conclusion, however, does not apply to the investment horizon of unsophisticated fundamentalists. Consider the case in which the market is populated by a group of chartists and a group of unsophisticated fundamentalists, described in Section 5.1. Figure 8 is a bifurcation diagram of price: we compute the set of price spanned by the system for different values of η_1, the parameter describing the time horizon of fundamentalists. As one can see, for small values of η_1, the dynamics stabilize around the fixed point. When η_1 increases, the system looses stability and goes to the region where dynamics is similar to Fig. 2. Hence, in the presence of two types of agents, the increase of the investment horizon of unsophisticated fundamentalists can have a detrimental effect on the stability of the fixed point. However, when η_1 increases further the fundamental value becomes stable again.

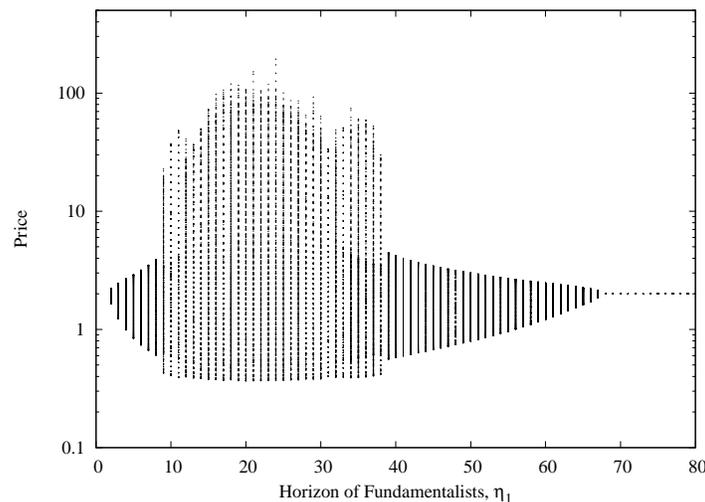This result comes quite unexpected. Indeed, almost by definition, the

Figure 8: Bifurcation diagram. The price support of a 1000 steps orbit (after a 1000 steps transient) is shown in the log-scale for different values of η_1 from 1 to 80. The values of the other parameters and the initial conditions are the same as in Fig. 2.

behavior of the fundamentalist traders, who buy the asset when its price is below the fundamental level and sell it when the price is above the fundamental, should bring stability to the system. The explanation has to be searched in the dependence of the fundamentalists' demand function (19) on the investment horizon captured by the coefficient B^{UF} defined in (20). Indeed, it turns out that this coefficient is a non-monotonic function of the investment horizon η. In Fig. 9 we report the increments in the value of B^{UF} when η increases from 1 to 2, 2 to 3, and 3 to 4, as a function of parameters r_f and θ. For large values of θ and small values of r_f these increments are negative, meaning that the optimal portfolio on longer time horizons contains a lower amount of risky security. However, we also observe that with increase of η, the region where $BUF(\eta+1,\theta) < BUF(\eta,\theta)$ shrinks, so that, eventually, B^{UF} becomes an increasing function of η for any values of $\theta > 0$ and r_f, see also Fig. 7. Even if the increase of horizons will eventually drive the system in the region of stability, for small enough values of η_1, *an increase of the horizon of fundamentalists can break the stability of the fixed point, while an increase of the horizon of chartists can not.*

Finally, since coefficient B^{SF} is an increasing function of η for any value of θ and r_f, an increase of the time horizons of the sophisticated fundamen-
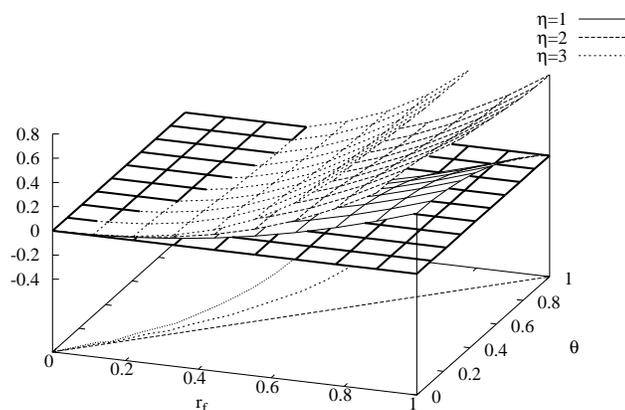
Figure 9: First difference $B^{\mathrm{UF}}(\eta + 1, \theta) - B^{\mathrm{UF}}(\eta, \theta)$ as a function of r_f and θ for different values of η. The intersections of the surfaces with the $z = 0$ plane are shown at the bottom of the graph.

talists always enlarges the stability domain of the "fundamental" fixed point, see Fig. 7.

The destabilizing effect of an increase of the unsophisticated fundamentalists' investment horizons observed in model with mixed population can be explained as a *substitution effect* of the demand structure. Indeed, the non-linear pricing equation (9) is defined in terms of the weighted demand of the different groups of traders. The stability/instability of the dynamics depends, therefore, not only on the total demand but also on the *relative* influence of fundamentalists (which contribute to the stabilization of the system) and chartists (who create the opposite effect). When the investment horizon of the unsophisticated fundamentalists increases (and not large enough), their total demand all other parameters being equal will *decrease*, as it is seen from Fig. 7. The overall effect is then equivalent to a reduction of the number of fundamentalists in the market: in this case the substitution effect plays a destabilizing role. On the other hand, if the fundamentalists form the long-term forecast for the variance according to (21), which increases with η_1 slower than (18), then the substitution effect plays a stabilizing role, increasing the relative weight of the fundamentalist traders in the market.

8   Conclusions

This paper extends the asset pricing model introduced in Bottazzi (2002) to incorporate heterogeneity in traders' investment horizons. Traders are described as mean-variance utility maximizers. They have different expectations about future returns and plan their investments over different time horizons.

Our first result is that the sole heterogeneity of investment horizons is not enough to destabilize the dynamics of prices. In other words, the market behavior is affected by the investment horizons of agents only when they possess heterogeneous beliefs about future prices. This result is relevant for the search of the minimal heterogeneity requirements sufficient to explain observed empirical regularity, e.g., excess volatility.

When a mixed populations of traders is considered, one observes the emergence of non-obvious effects of the length of the investment horizons on the market price. Indeed, we found that minor changes in the investment horizons of some sub-population of agents may lead to large qualitative changes in the dynamics of price. In general, however, the strength and nature with which a variation in investment horizons affects the market dynamics strongly depend on the ecology of agents considered. The overall effect can be traced back to the different behavior of agents individual demand functions when investment of different maturities are considered. These differences come, in turn, from the way in which different agents estimate long-run risk.

One obvious limit of the previous investigation resides in the simplifying assumption of a Walrasian market clearing. Indeed, it would be interesting to investigate our behavioral model in a different, non Walrasian, market architecture. As shown by Bottazzi et al. (2005) and Anufriev and Panchenko (2009) the trading protocol can have a big impact on the dynamics of price. For instance, LiCalzi and Pellizzari (2003) simulate an order-driven stock market which is operated through the book of orders. In their model all agents are fundamentalists and only differ with respect to their investment horizons. Nevertheless, they are able to generate time series with properties similar to the ones observed in real markets.

A second interesting extension would be the introduction of an endogenous mechanism in the selection of investment horizons. One possibility is to use the framework presented in Brock and Hommes (1998) and root the choice of the horizon inside the random utility theory. This extension could potentially enrich our model, in which horizons are exogenously given and fixed, and lead to an extension of the model in Osler (1995) where the dynamical aspect of the endogenous choice of horizons is absent.

APPENDIX

A Forecasts of sophisticated fundamentalists

In this Appendix, we find the time dynamics of the first two moments of the distribution of return. These moments are used by sophisticated fundamentalists as the forecast for the expected return and variance of the return. Recall that these agents think that the market always tends to correct the current price in the direction to the fundamental value. We can, first, describe such price behavior as mean-reverting stochastic process, then find the first two moments of the price and, finally, go back to the returns.

Imagine that at time t one is interested in the probability density $f(P, t+\tau)$ of asset price P after a time τ. Consider the Fokker-Plank equation describing the evolution of this density

$$\dot{f}(P, t+\tau) = -\frac{\partial}{\partial P}\left[a_1(P, t+\tau)\, f(P, t+\tau)\right] + \frac{1}{2}\frac{\partial^2}{\partial P^2}\left[a_2(P, t+\tau)\, f(P, t+\tau)\right], \quad (38)$$

where dot denotes the derivative with respect to time. Consistently with the assumptions about fundamentalist behavior, assume that the coefficient of diffusion a_2 is constant and denoted as σ^2, and that the drift describes a mean reversing process $a_1 = \tilde{\theta}(\bar{P} - P)$ whose strength is given by a positive parameter $\tilde{\theta}$.

Multiplying both sides of (38) by P and integrating with respect to the price, one obtains the differential equation for the first moment of the price distribution $m_1(t + \tau) = \int P\, f(P, t+\tau)dP$ which reads

$$\dot{m}_1(t + \tau) = \tilde{\theta}\,\bar{P} - \tilde{\theta}\,m_1(\tau).$$

Solving this equation with initial condition $m_1(t) = P_t$ one gets

$$m_1(t + \tau) = \bar{P} + (P_t - \bar{P})\,e^{-\tilde{\theta}\tau}. \quad (39)$$

Eq. (16) gives the condition for the correct definition of $\tilde{\theta}$, namely we have to impose that $m_1(t + 1) = \mathrm{E}_{t,f}[P_{t+1}]$. One has

$$\bar{P} + (P_t - \bar{P})\,e^{-\tilde{\theta}} = P_t + \theta\,(\bar{P} - P_t),$$

so that

$$\tilde{\theta} = -\ln(1 - \theta). \quad (40)$$

Eq. (39) describes the belief of the agent about the average price at time τ in the future. From $\mathrm{E}_t[\rho_{t+\eta}] = \mathrm{E}_t[P_{t+\eta}]/P_t - 1$ we can construct the belief about the average return: $\mathrm{E}_t[\rho_{t+\eta}] = (\bar{P}/P_t - 1)(1 - e^{-\tilde{\theta}\eta})$. Finally, substituting the expression for $\tilde{\theta}$ in (40) we get (17).

A similar procedure is repeated for the second moment. Multiplying both sides of (38) by P^2 and integrating with respect to P, one gets the differential equation for the second moment of price $m_2(t + \tau) = \int P^2 f(P, t + \tau) \, dP$ which reads

$$\dot{m}_2(t + \tau) = \sigma^2 + 2\tilde{\theta} \, \bar{P} \, m_1(t + \tau) - 2 \, \tilde{\theta} \, m_2(t + \tau) \, .$$

The solution with initial condition $m_2(t) = P_t^2$ becomes

$$m_2(t + \tau) = \frac{\sigma^2}{2\,\tilde{\theta}} + \bar{P}^2 + 2\bar{P}(P_t - \bar{P})e^{-\tilde{\theta}\tau} + \left(P_t^2 - \frac{\sigma^2}{2\,\tilde{\theta}} - \bar{P}^2 - 2\,\bar{P}\,(P_t - \bar{P}) \right) e^{-2\tilde{\theta}\tau} \, .$$

Now using the expression for the first two moments of the price distribution, we can compute the belief of the fundamentalists about the variance $V_{t,f}[\rho_{t,t+\eta}]$,

$$V_{t,f}[\rho_{t,t+\eta}] = \frac{1}{P_t^2} \Big(m_2(t + \eta) - m_1(t + \eta)^2 \Big) = \frac{\sigma^2}{2\tilde{\theta}P_t^2} \Big(1 - e^{-2\tilde{\theta}\eta} \Big) \, .$$

To get rid of the parameter σ^2 we assume that the one period ahead forecast of fundamentalists coincides with the forecast of chartists, i.e., it is given by the EWMA estimator z_{t-1}. Then

$$\sigma^2 = \frac{2\tilde{\theta}P_t^2}{1 - e^{-2\tilde{\theta}}} \, z_{t-1} \, ,$$

and, finally,

$$V_{t,f}[\rho_{t,t+\eta}] = \frac{1 - e^{-2\tilde{\theta}\eta}}{1 - e^{-2\tilde{\theta}}} \, z_{t-1} \, .$$

Substituting the expression for $\tilde{\theta}$ in (40) one gets (21).

Finally notice that the chartists forecasting rules (12) and (13) can be obtained following the same procedure but assuming constant drift and variance in the Fokker-Plank equation (38).

B   Proof of Proposition (4.1)

First of all notice that even if the function f in (27) is defined for positive arguments, it can be extended continuously to $z = 0$. In fact

$$\lim_{z \to 0} f(z) = \frac{s}{r} = \gamma \bar{P} \, .$$

Thus, system (27) is defined for any y and for $z \geq 0$.

In order to find the possible fixed points of the system assume that $P_{t+1} = P_t = P^*$. Using the second equation, this implies that $y_{t+1} = \lambda y_t$. If $\lambda > 0$, the

condition $y_{t+1} = y_t$ implies $y_t = 0$. The same applies to the equation for z_t. Then, substituting $y_t = z_t = 0$ in the first equation, one concludes that the system has only the fixed point $(\gamma\bar{P}, 0, 0)$, which corresponds to the fundamental price with zero forecasted variance.

The local stability of the point can be checked computing the Jacobian matrix. First, note that the derivative of function f reads

$$f'(z) = \frac{1}{z}\left(\frac{s}{\sqrt{r^2 + 4sz}} - f(z)\right).$$

This derivative can be extended to the point $z = 0$ continuously, and one has $f'(0) = -s^2/r^3$. Consequently the Jacobian matrix in $(\gamma\bar{P}, 0, 0)$ reads

$$\mathbf{J}(p, y, z)\Big|_{(\gamma\bar{P},0,0)} = \left\|\begin{array}{ccc} 0 & 0 & f'(0) \\ -(1-\lambda)\frac{f(0)}{(\gamma\bar{P})^2} & \lambda & (1-\lambda)\frac{f'(0)}{\gamma\bar{P}} \\ -2\lambda(1-\lambda)\frac{f(0)}{(\gamma\bar{P})^2}\tilde{h} & -2\lambda(1-\lambda)\tilde{h} & \lambda + 2\lambda(1-\lambda)\tilde{h}\frac{f'(0)}{(\gamma\bar{P})} \end{array}\right\|,$$

where $\tilde{h}$ stands for the value of the function $h(p, y, z) = \frac{f(z)}{p} - 1 - y$ in $(\gamma\bar{P}, 0, 0)$. Since $\tilde{h} = 0$, the Jacobian in the fixed point can be simplified to become

$$\mathbf{J}(p, y, z)\Big|_{(\gamma\bar{P},0,0)} = \left\|\begin{array}{ccc} 0 & 0 & -s^2/r^3 \\ -(1-\lambda)r/s & \lambda & -(1-\lambda)s/r^2 \\ 0 & 0 & 0 \end{array}\right\|.$$

The eigenvalues of $\mathbf{J}(p, y, z)$ are λ and 0 (with multiplicity two), which implies that the point $(\gamma\bar{P}, 0, 0)$ is locally asymptotically stable as far as $0 \leq \lambda < 1$.

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