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Wealth-driven competition in a speculative financial market: examples with maximizing agents

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This paper demonstrates how both the quantitative and qualitative results of a general, analytically tractable asset-pricing model in which heterogeneous agents behave consistently with a constant relative risk-aversion assumption can be applied to the special case of optimizing behaviour. The analysis of the asymptotic properties of the market is performed using a geometric approach that allows the visualization of all possible equilibria by means of a simple one-dimensional Equilibrium Market Curve. The case of linear (particularly, mean–variance) investment functions is thoroughly analysed. This analysis highlights the features that are specific to linear investment functions. As a consequence, some previous contributions of the agent-based literature are generalized.

Keywords: Asset pricing model; CRRA framework; Equilibrium market curve; Expected utility maximization; Mean-variance optimization; Linear investment functions

1. Introduction

In recent years a number of theoretical models exploring the consequences of the heterogeneity of traders for the aggregate price dynamics of a speculative financial market have been developed. Among many examples, let us mention the models of Day and Huang (1990), DeLong *et al*. (1991), Chiarella (1992), Lux (1995), Brock and Hommes (1998) and Chiarella and He (2001). These and other 'heterogeneous agent models' (HAMs) have recently been reviewed by Hommes (2006). Within the 'agent-based' literature, HAMs can be seen as an important branch of study supplementary to the numerous simulation models. Indeed, one of the problems with the simulation approach is that the systematic analysis of such models is made practically impossible by the enormous number of degrees of freedom. It is usually not clear which assumptions are responsible for the patterns generated and, as a result, the robustness of the models is difficult to investigate. HAMs have appeared as a response to this problem and, consequently, are built in such a way as to make analytic investigation possible. It is not surprising, therefore, that heterogeneous agent models usually incorporate only a few types of agents who differ in the ways they predict the future price but are homogeneous in all other respects, i.e. in the functional form of demand, available information, etc.

Even if the analytic models have already answered a lot of theoretical questions concerning the consequences of behavioural heterogeneity for market dynamics, they suffer from some important drawbacks. First, most of the contributions are built within the constant absolute risk-aversion (CARA) framework, that is under the assumption that demand is independent of wealth. This leads to simplification in the analysis, because, otherwise, the wealth of each individual portfolio along the evolution of the economy would have to be taken into account. However, this assumption is rather unrealistic if compared with other possible behavioural specifications, e.g. with constant relative risk aversion (CRRA) (see Levy *et al*. (2000) or Campbell and Viceira (2002) for a discussion).

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Second, the majority of the models consider only a few types (or classes) of behaviour, thus substantially reducing the realistic level of heterogeneity.† Third, the tests for the robustness of the results with respect to the change of simple behavioural assumptions are very difficult to perform inside such models. For example, in order to understand the consequences of the entry of an agent with a new type of behaviour in the market, one has to analyse a completely new model from scratch. Summarizing, one can say that, at this moment, HAMs lack a general framework, flexible enough to incorporate different realistic agents' specifications.

An important step in the direction of a general framework has been made by Anufriev *et al.* (2006) and Anufriev and Bottazzi (2006), where some analytic results are obtained for a market populated by an arbitrarily large number of technical traders whose possible demand functions belong to a relatively large set. The only imposed restriction on the individual demand functions is that they have to be proportional to current wealth. This requirement is consistent with the *constant relative risk-aversion* (CRRA) framework. Consequently, the price and agents' wealth are determined at the same time and both price and wealth dynamics are intertwined. To model the agents' behaviour, Anufriev and Bottazzi (2006) introduce deterministic *investment functions* which map the past history of returns into the fraction of wealth which is invested in the risky security. These investment functions are left unspecified, so that the obtained results are very general.

The purpose of the current paper is to provide an illustration of how this general, analytically tractable agent-based model can be applied to important particular classes of investment behaviour. Our main interest is focused on the functions which can be derived from the optimization principle and, therefore, can be considered as 'rational'. According to conventional economic wisdom such optimizing behaviour is a characteristic of the majority of the agents in financial markets, and therefore corresponding investment functions deserve a special analysis. We consider the investment functions derived from two types of rational choice procedures, expected utility (EU) maximization and mean–variance utility (MVU) maximization.

It is a well-known problem of the EU maximization framework with CRRA traders that the resulting demand functions cannot be computed explicitly. In order to overcome this obstacle a geometric tool called the 'Equilibrium Market Curve' will be used. It allows one to characterize both the location of all possible equilibria and (partially) the conditions of their stability *independent* of the specification of the traders' demands. In this way we obtain some predictions of equilibrium dynamics with EU maximizers even without explicit knowledge about their investment functions.

In contrast to the EU framework, the solution of the MVU optimization problem can be derived explicitly. The resulting demand depends on the agent's expectations about the mean and variance of the return for the next period. It turns out that, for some large class of these expectations, the investment functions become 'linear' in the sense which will be clarified later. Since different types of expectations can still lead to different investment functions, we keep the discussion as general as possible and investigate the dynamics in the market with 'linear' investment functions. In particular, we demonstrate that the phenomenon of multiple stable equilibria cannot emerge in such markets. This can be a limitation of the 'rational' framework with respect to the general case.

The analysis of the linear investment functions brings us to another goal of this paper. We show that one of the first analytical models developed in the CRRA framework, namely the model of Chiarella and He (2001), can be easily understood and generalized, when considered within the general framework of Anufriev and Bottazzi (2006). As a direct consequence, we can discuss the validity and limits of the 'quasi-optimal selection principle' originally formulated by Chiarella and He. We show that this principle is a consequence of particular market behaviours and it does not hold in general.

The rest of the paper is organized as follows. In the next section a stochastic model of a speculative market is introduced. In section 3 the steady states of its deterministic part are derived and characterized geometrically. The immediate applications for two special cases of investment behaviour are discussed. In section 4 the question of stability of the steady states is addressed with discussion of consequences for two special cases. The model of Chiarella and He is reconsidered in section 5 and some final remarks are given in section 6.

2. A dynamic model for asset price and agents' wealth

In this section we present the general analytic model of a market in which the individual demand functions for the risky asset are proportional to the agents' wealth.

2.1. General setup

Consider a simple pure exchange economy, populated by a fixed number N of traders, where trading activities take place in discrete time. The economy is composed of a riskless asset yielding in each period a constant interest rate $r_f > 0$ and a risky asset paying a random dividend D_t at the beginning of each period t. The riskless asset is considered to be the numéraire of the economy and its price is fixed to 1. The ex-dividend price P_t of the risky asset is determined at each period through a

†For instance, DeLong *et al.* (1991) consider two types of investors, the model of Day and Huang (1990) is populated by three types of traders, while Brock and Hommes (1998) provide a number of examples with two, three and four different types. One recent exception from this rule is the model of Brock *et al.* (2005) where the low-dimensional *Large Type Limit* with the number of types converging to infinity is introduced.

market-clearing condition, where the outside supply of the asset is constant and normalized to 1.

Let $x_{t,n}$ be the fraction of the wealth $W_{t,n}$ which, at time t, agent n ($n \in \{1, \ldots, N\}$) invests in the risky asset. The evolution of the economy is described by the system containing the individual wealth dynamics and the market-clearing condition

$$W_{t,n} = (1 - x_{t-1,n})W_{t-1,n}(1 + r_f) + \frac{x_{t-1,n}W_{t-1,n}}{P_{t-1}}(P_t + D_t),$$
$$P_t = \sum_{n=1}^{N} x_{t,n}W_{t,n}. \tag{1}$$

Assume that the individual demand for the risky asset is proportional to the current wealth, so that $x_{t,n}$ is independent of $W_{t,n}$. Price and wealth are determined simultaneously in this case. Hence, one has to solve the system (1) in order to obtain the evolution of P_t and $W_{t,n}$ in explicit form.

Introduce the price return $r_{t+1} = P_{t+1}/P_t - 1$, the dividend yield $y_{t+1} = D_{t+1}/P_t$, and the wealth share of agent n in the total wealth $\varphi_{t,n} = W_{t,n}/\sum_m W_{t,m}$. With a bit of algebra, one can show that, under suitable conditions,† the implicit dynamics (1) is equivalent to the following system of the return and wealth shares:

$$r_{t+1} = r_f + \frac{(1 + r_f)\sum_n (x_{t+1,n} - x_{t,n})\varphi_{t,n} + y_{t+1}\sum_n x_{t,n}x_{t+1,n}\varphi_{t,n}}{\sum_n x_{t,n}(1 - x_{t+1,n})\varphi_{t,n}}, \tag{2}$$

$$\varphi_{t+1,n} = \varphi_{t,n}\frac{1 + r_f + (r_{t+1} + y_{t+1} - r_f)x_{t,n}}{1 + r_f + (r_{t+1} + y_{t+1} - r_f)\sum_m x_{t,m}\varphi_{t,m}},$$
$$\forall n \in \{1, \ldots, N\}. \tag{3}$$

Notice that the dynamics in (2) and (3) do not depend on the price level directly, but, instead, are defined in terms of price return and dividend yield. In compliance with intuition, in the CRRA framework, where agents' demands are growing with their wealth, the equilibria can be identified as states of steady expansion (or contraction) of the economy.

Concerning the stochastic (due to random dividend payment D_t) yield process $\{y_t\}$ we make the following assumption.

Assumption 2.1: *The dividend yields y_t are i.i.d. random variables obtained from a common distribution with positive support and mean value $\bar{y} \in (0, 1)$.*

This assumption is common to a number of studies in the literature (see, for example, Chiarella and He (2001, 2002) and Anufriev *et al.* (2006)), and also roughly consistent with real data. We assume that the structure of the yield process is known to everybody. Consequently, the information set available to traders at round t reduces to the sequence of past realized returns $\mathcal{I}_{t-1} = \{r_{t-1}, r_{t-2}, \ldots\}$.

2.2. *Behaviour of traders*

To close the dynamical system (2) and (3) we only need to specify the set of investment shares $\{x_{t,n}\}$. Let us first discuss a number of possible ways to do so.

2.2.1. Maximization of mean–variance utility.

One way to derive the investment choice from the standard economic framework is to consider the mean–variance (MV) problem

$$\max_{x_t}\left\{\mathrm{E}_{t-1}[W_{t+1}] - \frac{\beta}{2W_t}\mathrm{V}_{t-1}[W_{t+1}]\right\}, \quad \text{where } \beta > 0. \tag{4}$$

The wealth evolution of an agent is given by $W_{t+1} = (1 + r_f)W_t + x_t W_t(r_{t+1} + y_{t+1} - r_f)$ and depends on the unknown at time t total return $r_{t+1} + y_{t+1}$. Thus, an agent has to form expectations $\mathrm{E}_{t-1}[W_{t+1}]$ and $\mathrm{V}_{t-1}[W_{t+1}]$ about the first two moments of his future wealth on the basis of the information set $\mathcal{I}_{t-1}$ available before period t.

In contrast to the standard MV framework, the coefficient measuring the sensitivity of the agent's utility to the risk decreases with wealth in (4). Therefore, the risk aversion is not constant but a decreasing function of wealth. This is consistent with experimental studies (see, for example, Kroll *et al.* (1988)).

A simple computation shows that the solution of (4) is given by

$$x_t = \frac{1}{\beta}\frac{\mathrm{E}_{t-1}[r_{t+1} + y_{t+1} - r_f]}{\mathrm{V}_{t-1}[r_{t+1} + y_{t+1}]}, \tag{5}$$

and does not depend on the current wealth. $\mathrm{E}_{t-1}[r_{t+1} + y_{t+1} - r_f]$ and $\mathrm{V}_{t-1}[r_{t+1} + y_{t+1}]$ are the agent's expectations concerning the excess return and variance, respectively.

Alternatively, one can obtain (5) as a solution of the MV problem written in terms of the agents' return $\rho_{t+1} = r_f + x_t(r_{t+1} + y_{t+1} - r_f)$. Maximization of $\mathrm{E}_{t-1}[\rho_{t+1}] - \beta\mathrm{V}_{t-1}[\rho_{t+1}]/2$ instead of (4) stresses another empirical finding that agents are concerned about the relative change of wealth and not about the level of their final wealth (Kahneman and Tversky 1979).

2.2.2. Maximization of expected utility.

A more sophisticated way to derive the demand is to maximize an expected utility (EU). Consider the *power* utility function of wealth

$$U(W; \gamma) = \frac{W^{1-\gamma} - 1}{1 - \gamma}, \quad \text{where } \gamma > 0. \tag{6}$$

It is straightforward to see that the solution $x_t^* = \arg\max \mathrm{E}[U(W_{t+1}; \gamma)]$ is independent of the agent's wealth W_t. On the other hand, this solution depends on

†These are the conditions to guarantee that price is positive (Anufriev *et al.* 2006).

the agent's perception of the *distribution* of the total return $r_{t+1}+y_{t+1}$. Unfortunately, an explicit functional form of this solution cannot be derived for all reasonable distributions.

Another possibility is to consider the EU maximization problem with *exponential* utility function $U(\rho_{t+1};\beta)=-\exp(-\beta\rho_{t+1}/2)$ of the agents' return ρ_{t+1}. One can show that, if agents perceive a normal distribution of the total asset return with expected value $\mathrm{E}_{t-1}[r_{t+1}+y_{t+1}]$ and variance $\mathrm{V}_{t-1}[r_{t+1}+y_{t+1}]$, then the solution of the corresponding EU maximization coincides with (5).

2.2.3. Generalization: investment function. Let us compare these examples of the agents' behaviour. In all the cases discussed above the optimal share of the wealth invested in the risky asset is independent of the contemporaneous variables, current wealth and current price.

In some cases this share x_t is given explicitly by (5) and depends on the agent's beliefs about expected excess return and its variance. These beliefs, in turn, are based either upon the commonly available distribution of the dividend yield, or upon the previous return history, or both. Essentially, the share x_t is evolving as a function of previous information.

For the EU maximization with power utility the optimal share x_t^* is unknown in explicit form. But if the agent perceives some distribution (e.g. log-normal) for the return $r_{t+1}+y_{t+1}$, he will have to update the parameters of this distribution on the basis of past information. Ultimately, the optimal share is again some function of the information set.

This similarity between different examples suggests the following assumption.

Assumption 2.2: *For each agent n there exists a finite memory time span L* (*which, without loss of generality, can be assumed to be the same for all the agents*), *and differentiable investment function f_n which maps the present information set consisting of the past L available returns into his investment share*:

$$x_{t,n}=f_n(r_{t-1},\ldots,r_{t-L}). \tag{7}$$

The function f_n on the right-hand side of (7) gives a complete description of the investment decision of agent n. The knowledge about the fundamental dividend process is not included in the information set but is embedded in the function f_n itself.

Assumption 2.1 and equations (2), (3) and (7) define a stochastic dynamical system. To obtain insight into the dynamics of the system we will follow the strategy common in this type of literature (see, for example, Brock and Hommes (1998), Chiarella and He (2001) and Chiarella *et al.* (2006)) and substitute the realization of the yield process by its mean value $\bar{y}$. The corresponding deterministic system reads

$$\begin{aligned} x_{t+1,n}&=f_n(r_t,\ldots,r_{t-L+1}),\\ r_{t+1}&=r_f+\frac{(1+r_f)\sum_n(x_{t+1,n}-x_{t,n})\varphi_{t,n}+\bar{y}\sum_n x_{t,n}x_{t+1,n}\varphi_{t,n}}{\sum_n x_{t,n}(1-x_{t+1,n})\varphi_{t,n}},\\ \varphi_{t+1,n}&=\varphi_{t,n}\frac{1+r_f+(r_{t+1}+\bar{y}-r_f)x_{t,n}}{1+r_f+(r_{t+1}+\bar{y}-r_f)\sum_m x_{t,m}\varphi_{t,m}}. \end{aligned} \tag{8}$$

Before proceeding further let us establish the relation of the model presented so far with some previous contributions. Assumption 2.2 makes our framework relatively rich in terms of possible agents' behaviours. It includes the entire model of Chiarella and He (2001), where the agents' investment shares are given by (5). As another special case the model includes the behaviours considered in the simulation models of Levy *et al.* (1996) and Zschischang and Lux (2001), where agents maximize the expected power utility. The model also generalizes the contribution of Anufriev *et al.* (2006), where the agents' demands are given as arbitrary functions of the exponentially weighted moving averages of past returns.

At the same time, some standard approaches to the modeling of the agents' behaviours are not consistent with assumption 2.2. For instance, in the models of Brock and Hommes (1998), LeBaron *et al.* (1999) and Brock *et al.* (2005) the agents have constant absolute risk-aversion demand functions, where $x_{t,n}$ depends on the contemporaneous wealth $W_{t,n}$.

3. Equilibria in the deterministic skeleton

Below we characterize the fixed points of system (8), first for the general investment functions and then for the special cases. The next section is devoted to the stability analysis of the steady states. These steps allow us to obtain insight into the long-run behaviour of system (8).

3.1. General result and the Equilibrium Market Curve

A fixed point of the skeleton is composed of the price return r^*, the equilibrium investment shares of all the agents x_n^*, and the relative wealth shares of the agents φ_n^*. The agents with $\varphi_n^*\neq 0$ are called *survivors*.

Two observations help to characterize the fixed points of the multi-dimensional system (8). First, the last equation in (8) gives the evolution of the agent's relative wealth. Similar to the replicator dynamics known from evolutionary biology, it provides the selection mechanism. If an agent obtain return higher (lower) than the average return, his relative wealth grows (declines). Due to such selection, in the fixed point all survivors must have the same return. The return of agent n is equal to $r_f+x_n^*(r^*+\bar{y}-r_f)$, so the investment shares of all survivors must also be the same. The only exceptions are the equilibria with $r^*+\bar{y}=r_f$ where both assets are equivalent in terms of return, and all the agents earn the same return independent of their investment shares. In such equilibria the wealth shares of all the agents are constant over time.

Second, the equilibrium return and investment shares are interrelated through the first two equations of (8). These equations provide, in a sense, consistency conditions between the agents' behaviour and the aggregate dynamics. They are considerably simplified in the fixed point due to the constant return history and consequences of the wealth evolution as discussed in the previous paragraph. Going through some simple algebra, one obtains the following result.

Proposition 3.1: *Let* $\mathbf{x}^*$ *be a fixed point of the system* (8). *Then*

$$x_n^* = f_n(r^*, \ldots, r^*), \quad \forall n \in \{1, \ldots, N\}, \tag{9}$$

and the following two cases are possible.

(i) *Equity premium* (EP) *equilibria. In* $\mathbf{x}^*$ *there are* k *survivors* ($1 \le k \le N$) *investing the same share* $x^*_{1\diamond k}$. *The wealth shares of survivors are arbitrary numbers summing to* 1, *while the remaining agents have zero wealth shares. The equilibrium return* r^* *satisfies*

$$r^* = r_f + \bar{y}\,\frac{x^*_{1\diamond k}}{1 - x^*_{1\diamond k}}. \tag{10}$$

(ii) *No equity premium* (*NEP*) *equilibria. In* $\mathbf{x}^*$ *the equilibrium return* $r^* = r_f - \bar{y}$. *The investment and wealth shares of the agents satisfy*

$$\sum_{n=1}^{N} x_n^* \varphi_n^* = 0 \quad \text{and} \quad \sum_{n=1}^{N} \varphi_n^* = 1. \tag{11}$$

Proposition 3.1 shows that two types of equilibria are possible. If two assets give the same return, there is no equity premium of the risky asset. The selection mechanism does not work, but the consistency conditions imply the constraint (11). In all other equilibria there is an equity premium, positive or negative. Our main focus will be on such 'equity premium' equilibria. To give them a geometric representation we introduce a special geometric locus.†

Definition 3.2: The Equilibrium Market Curve (EMC) is the function $l(r)$ defined as

$$l(r) = \frac{r - r_f}{r + \bar{y} - r_f}, \quad \text{for } r > -1.$$

Equation (10) can now be written as $x^*_{1\diamond k} = l(r^*)$. Together with (9), it implies that, for every survivor,

$$l(r^*) = f_n(\underbrace{r^*, \ldots, r^*}_{L}). \tag{12}$$

Thus, the equilibrium return, r^*, and the equilibrium investment share of survivors, $x^*_{1\diamond k}$, are simultaneously determined at an intersection point of two curves. One

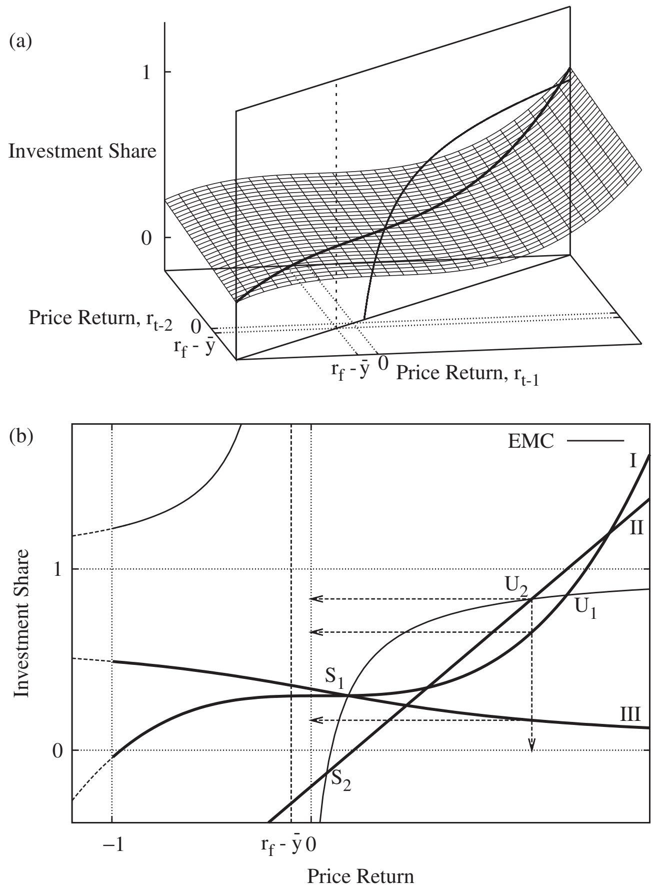

Figure 1. Panel (a): an example of a two-dimensional investment function $f(r_{t-1}, r_{t-2})$. The symmetrization of the investment function is shown as the thick curve, and the EMC is shown as the thin curve in the diagonal cross section. Panel (b): the intersections of the EMC and different (symmetrizations of the) investment functions. Each of the points S_2, U_1 and U_2 defines one equilibrium of system (8) with one survivor. Point S_1 defines infinitely many equilibria with two survivors.

curve is the EMC, which depends on the risk-free interest rate and the dividend yield and is independent of the agents' behaviour. All individual behaviour relevant for the equilibrium dynamics is encompassed in the second curve, which is the graph of the investment function under the constraint $r_{t-1} = \cdots = r_{t-L}$. However complex the agents' behaviour might be, it is only its restriction on the constant return history that matters for the characterization of equilibrium. This almost tautological statement has important implications for the geometric characterization of equilibria. The decisions of an agent can be represented as a graph of the investment function defined in (7). In general, this graph is an L-dimensional surface, as illustrated in panel (a) of figure 1 for the case $L = 2$. However, all equilibria can be found in a diagonal cross section of this surface. Starting with an arbitrary investment function we give the following definition.

Definition 3.3: The symmetrization of the investment function f is a cross section of the graph of

†Introduced by Anufriev *et al.* (2006) in a slightly different notation, this curve was first called the 'Equilibrium Market Line'.

$f(r_{t-1}, \ldots, r_{t-L})$ by the diagonal hyperplane $r_{t-1} = \cdots = r_{t-L}$.

The surface in panel (a) of figure 1 is intersected by the diagonal plane and the resulting symmetrization is shown as a thick curve. From (12) it follows that all possible equilibria are intersections of the symmetrization with the EMC (a thin hyperbolic curve in the cross section). In the following, by a slight abuse of language, the symmetrizations of the investment function will simply be referred to as 'investment functions'.

In panel (b) of figure 1 the EMC is shown together with three investment functions† marked I, II and III. Each of them is, indeed, a symmetrization of some multi-dimensional investment function. For instance, the nonlinear curve of agent I is the same as in the cross section of panel (a). All four intersections with the EMC represent equilibria in such a market with three agents. Consider, first, point U_2. Agent II with the linear investment function is the only survivor. The abscissa of U_2 gives the equilibrium return, the ordinate gives the investment share of the survivor. The investment shares of the two other agents are the values of their investment functions for the equilibrium return (see the arrows and equation (9)). Analogously, in equilibrium U_1 agent I is the only survivor and the equilibrium return is the highest possible for such a market. In S_2, agent II is the survivor and the return is the smallest possible. Finally, consider equilibrium S_1. Since two investment functions intersect the EMC at the same point, there are two survivors here. Agent II, who does not survive, has $\varphi_2^* = 0$. But the only restriction on the two other wealth shares is $\varphi_1^* + \varphi_3^* = 1$. Therefore, a single point S_1 represents, strictly speaking, infinitely many equilibria of the system (8).

The reader may suspect that the latter situation with the coexistence of different agents in the EP equilibrium is exceptional, since the only way for this to occur is that two or more investment functions intersect the EMC at the same point. Recall, however, that the functions which we see on the plot are the one-dimensional symmetrizations of the investment functions. Imagine a situation where the agents have the same demand function which depends on the expected return. Agent n take an average of past L_n returns as predictor. Since in the equilibrium the span does not matter, the investment functions of different agents will have the same symmetrization. In the plot with the EMC they will be indistinguishable and any intersection with the EMC will define infinitely many equilibria with arbitrary division of wealth between agents. On the other hand, these equilibria generate the same price dynamics since the return is the same in all of them.

To summarize, we investigated the questions of the existence and location for the fixed points of deterministic system (8). It is no surprise that, with general specification of the investment function, we do not obtain a definite answer. However, we showed that all possible fixed points belong to the one-dimensional Equilibrium Market Curve whose shape is completely determined by the exogenous parameters of the system. The EMC in definition 3.2 is defined only for $r > 1$, which guarantees the positiveness of price. In such case we call the equilibrium *feasible*. Now we restrict the generality of investment functions and use the EMC for some special cases.

3.2. Equilibria for linear investment functions

Assume that the symmetrization of the investment function f is a *linear* function. This symmetrization will simply be called a 'linear investment function'. The linear function is a special but not peculiar case. Consider, for instance, the investment share (5) derived from the MV optimization. An agent knows that the mean and variance of the yield are constant. Estimating the first two moments of the price return as sample moments on the basis of the past L observations, she will obtain $\mathrm{E}_{t-1}[r_{t+1}] = r$ and $\mathrm{V}_{t-1}[r_{t+1}] = 0$, and her investment function will obviously be linear.

Let us now apply the equilibrium analysis of section 3.1 for such linear investment functions. We consider only the equilibria with equity premium. In this case from proposition 3.1 it is clear that the properties of all multi-agent equilibria can be easily understood by studying the single-agent case. The geometric plot of the EMC suggests that there can exist at most two equilibria for any linear investment function.

To formalize this, introduce the following parameterization of the investment function:

$$f(r, \ldots, r) = (A + 1) + B(r + \bar{y} - r_f), \tag{13}$$

where B is the slope of the function, and $A + 1$ is an intersection with the vertical asymptote of the EMC. This parameterization is illustrated in panel (a) of figure 2.

The following statement describes all the equilibrium possibilities for the market with one linear investment function.

Proposition 3.4: *Consider equilibria of the market with a single agent possessing the investment function with linear symmetrization* (13). *Then the following two cases are possible.*

(i) *Constant function:* $B = 0$. *For* $A = 0$ *there are no equilibria. If* $A \neq 0$ *there exists one equilibrium with return*

$$r^* = r_f - \frac{\bar{y}}{A} - \bar{y} \tag{14}$$

which is feasible, i.e. generates positive price, when $A < 0$ *and when* $A > A_F \equiv \bar{y}/(1 + r_f - \bar{y})$.

†The illustration corresponds to the case when $r_f < \bar{y}$. This is consistent with the real data collected by Robert Shiller and available at http://www.econ.yale.edu/~shiller/data.htm. For the period from 1871 to 2005 the average real one-year interest rate was 0.029 and the average dividend yield was 0.044.

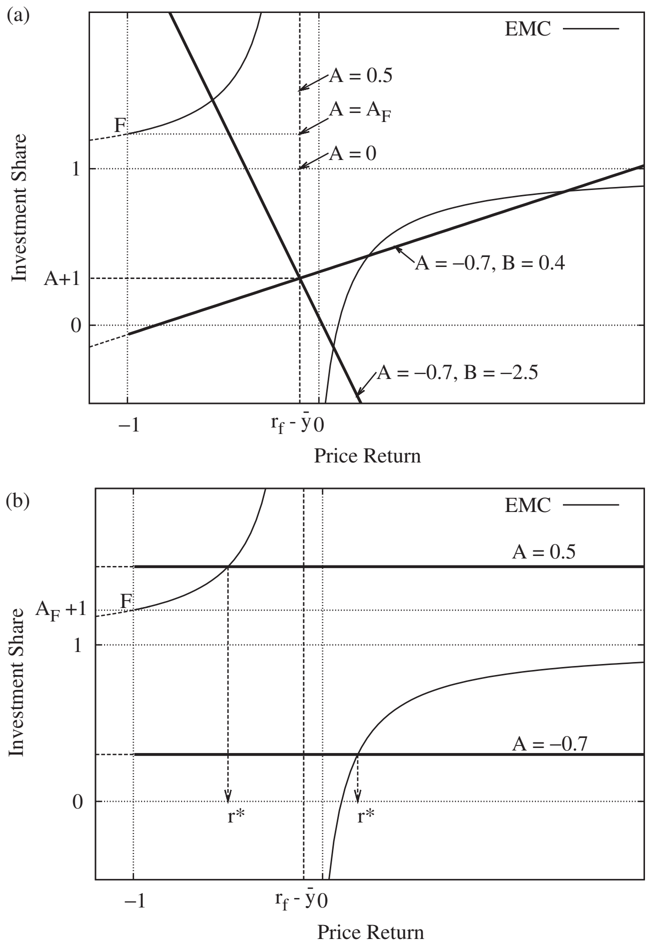

Figure 2. Panel (a): examples of the linear investment functions (13) for different parameter pairs (A, B). Panel (b): equilibria for constant investment functions.

(ii) *Non-constant function*: $B \neq 0$. *Set* $D = A^2 - 4B\bar{y}$. *If* $D < 0$, *then there are no equilibria. Otherwise, when* $D \geq 0$, *there are two equilibria (coinciding when* $D = 0$*) with the following returns*:

$$r_1^* = r_f + \frac{-A - \sqrt{A^2 - 4B\bar{y}}}{2B} - \bar{y},$$
$$r_2^* = r_f + \frac{-A + \sqrt{A^2 - 4B\bar{y}}}{2B} - \bar{y}. \qquad (15)$$

The equilibrium is feasible if the return exceeds -1.

Proof: See Appendix A. □

This proposition provides all possible equilibrium values of the return for different linear investment functions (13). When $B = 0$ the agent's investment does not depend on the past information and his function represents the horizontal line as shown in panel (b) of figure 2. For A positive the unique equilibrium belongs to the left branch of the EMC and generates negative equity premium. This equilibrium is feasible only if $A > A_F$, otherwise the prices are negative. For A negative the equilibrium belongs to the right branch of the EMC, so that there is a positive equity premium.

When $B \neq 0$ one can distinguish between two cases (see panel (a) of figure 2). If the investment function is

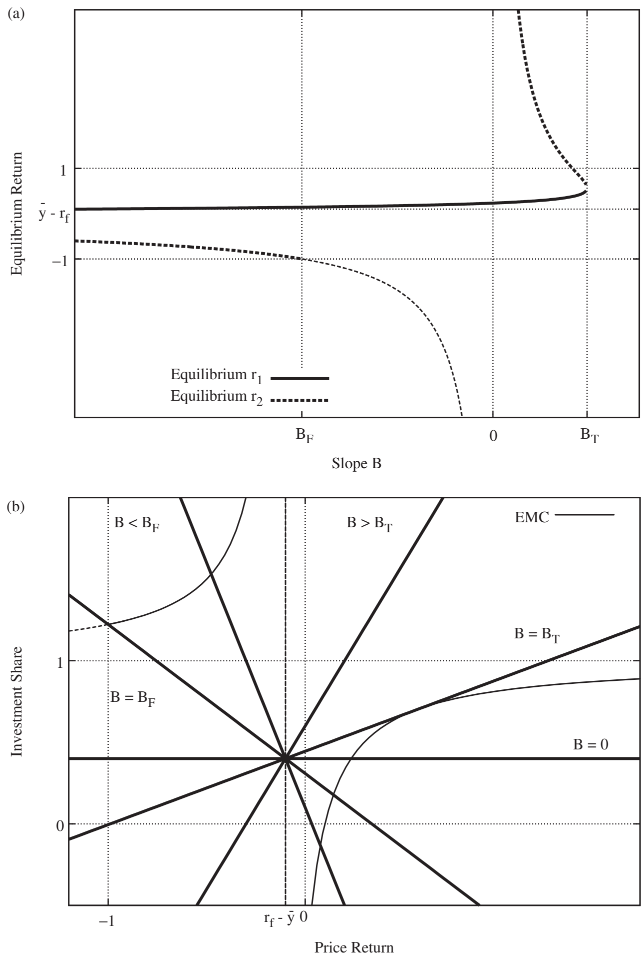

Figure 3. Equilibria r_1^* and r_2^* computed in proposition 3.4 as functions of the slope B of the linear investment function for $A = -0.6$ (panel (a)), and their movement as a result of the rotation of the investment function on the EMC plot (panel (b)).

decreasing, so that $B < 0$, it is always the case that $D > 0$ and, therefore, two equilibria exist. From (15) it follows that equilibrium r_1^* belongs to the left branch of the EMC, while equilibrium r_2^* belongs to the right branch of the EMC. In the opposite case, when $B > 0$, the investment function increases and can have zero, one or two equilibria. In the latter situation, $r_1^* < r_2^*$ and both equilibria belong to the left (right) branch of the EMC when $A > 0$ ($A < 0$).

Notice how easily one can do comparative statics exercises with the aid of such plots. For example, let us fix A and change the slope B of the investment function. In panel (a) of figure 3 we draw the equilibria derived in proposition 3.4 for $A = -0.6$ as functions of B. But these plots are obvious even without computations if one uses the EMC! Indeed, look at panel (b) of the same figure. Start with a vertical investment function and rotate it in the counterclockwise direction, so that B increases from $-\infty$. First, there exist two equilibria with returns close to $r_f - \bar{y}$. In one of them the equity premium is positive and the return increases. In the second equilibrium the equity premium is negative, the return decreases and becomes equal to -1 at some value $B = B_F$. This equilibrium is not

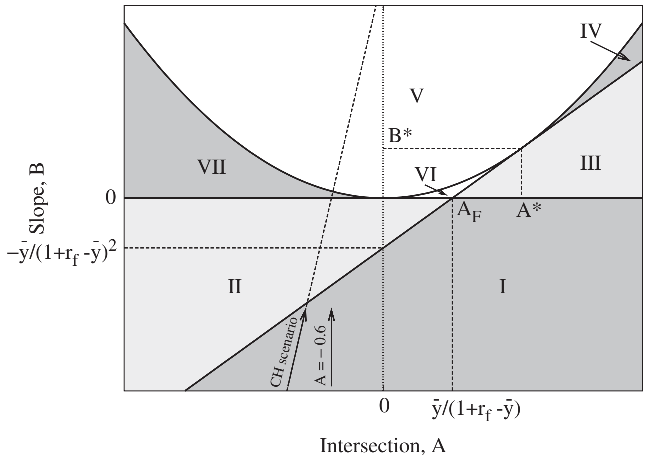

Figure 4. Stratification of the parameter space (A, B) according to the number of feasible equilibria. The dark grey (light grey, white) area represents the parameters for which there are two (one, zero) equilibria. See text for an explanation of the regions marked by Roman numerals. The two specific examples illustrated in figures 3 and 5 correspond to movement along the directions shown by the arrows.

feasible for $B > B_F$. Increase B further until zero value. Now the investment function is horizontal with a unique equilibrium. Further rotation leads to the emergence of the second equilibrium in the right branch of the EMC. The return in this equilibrium decreases. Finally, at some value $B = B_T$ the investment function is tangent to the EMC and two equilibria coincide. For larger B there are no equilibria.

Figure 4 shows the stratification of the parameter space (A, B) according to the number of different equilibria. For the parameter pairs from the white area there are no feasible equilibria, for those pairs which belong to the light grey area, only one such equilibrium exists, and, finally, if parameters belong to the dark grey area there exist two different equilibria. Three loci are important for the stratification of the parameter space. They are shown by the thick curves and divide the space into seven different regions marked by the Roman numerals. The first locus is a horizontal straight line corresponding to $B = 0$. In this case the investment function is horizontal and there exists one equilibrium. Any change of B leads to the appearance of the second equilibrium which can be with negative prices (unfeasible), though. The curve with parabolic shape contains the points with $A^2 = 4B\bar{y}$, i.e. those parameters for which the equilibrium is unique due to the tangency between the EMC and the investment function (13), like in figure 3 for $B = B_T$. Thus, this parabola separates the parameters for which there are no equilibria (region V) from those for which two equilibria exist. This locus is called the 'tangency curve'. Finally, the third locus corresponds to the parameter pairs for which the linear investment function passes through point F of the upper-left branch of the EMC, like in figure 3 for $B = B_F$. These pairs lie on the increasing line $A = \bar{y}/(1 + r_f - \bar{y}) - B(1 + r_f - \bar{y})$. With the crossing of this locus, which we call the 'feasibility curve', one feasible equilibrium is lost.† If $B < 0$, the equilibrium on the upper-left branch of the EMC disappears with decreasing A, so that two regions I and II are determined. If $B > 0$ and $A < 0$, then, as mentioned above, both equilibria (if they exist) belong to the lower-right branch of the EMC and both are feasible, so that area VII is determined. Finally, if both A and B are positive, let us denote as (A^*, B^*) the parameter pair defining the investment function which passes through F and, at the same time, is tangent to the EMC. In region III, only the equilibrium with the smallest return r_1^* is feasible. When A decreases there are two possible options: either r_1^* also becomes infeasible or r_2^* becomes feasible. From the EMC plot it is easy to see that the first case happens for $B < B^*$, i.e. when at point F the investment function is flatter than the tangency line. In this case, from region III we move to region VI. The second case happens when $B > B^*$ and we move from region III to IV.

3.2.1. An example of a linear investment function. Following Chiarella and He (2001), let us consider an agent with constant relative risk aversion β whose investment share is given by (5). Assume that his expectations are as follows:

$$\mathrm{E}_{t-1}[r_{t+1} + y_{t+1}] = r_f + \delta + dm_t, \qquad (16)$$

$$\mathrm{V}_{t-1}[r_{t+1} + y_{t+1}] = \sigma^2(1 + b(1 - (1 + v_t)^{-\xi})), \qquad (17)$$

where m_t and v_t denote the sample estimates of the average return and its variance computed as equally weighted averages of the previous L observations

$$m_t = \frac{1}{L}\sum_{k=1}^{L}(r_{t-k} + y_{t-k}) \quad \text{and} \quad v_t = \frac{1}{L}\sum_{k=1}^{L}(r_{t-k} + y_{t-k} - m_t)^2.$$

The expression for the variance is justified by Franke and Sethi (1998), however this choice and, in particular, positive parameters b and ξ turn out to be irrelevant for the equilibrium analysis. The specification of the expected conditional return (16) is important, however. It is defined as the risk-free rate r_f plus the excess return. The latter is composed of a constant component representing a risk premium, $\delta \geq 0$, and a variable component, dm_t. The parameter d represents the way in which agents react to variations in the history of realized returns and can be used to distinguish between different classes of investors. A trader with $d = 0$ will ignore past realized returns and, consequently, can be thought of as a *fundamentalist*. If $d > 0$ the agent can be considered a *trend-follower*, if $d < 0$ he can be considered a *contrarian*.

†At this point the reader is highly encouraged to follow the discussion drawing the different possible mutual locations of the linear investment function and the EMC.

Direct substitutions of (16) and (17) into (5) gives the investment function

$$f^{\mathrm{CH}}(r_{t-1},\ldots,r_{t-L}) = \frac{\bar{\delta}+\bar{d}m_t}{1+b(1-(1+v_t)^{-\xi})},$$
$$\text{with } \bar{\delta}=\frac{\delta}{\beta\sigma^2}, \quad \bar{d}=\frac{d}{\beta\sigma^2}. \quad (18)$$

In the equilibrium of the deterministic skeleton, $m_t=\bar{y}+r^*$ and $v_t=0$. Therefore, we are dealing with linear symmetrization

$$f^{\mathrm{CH}}(r,\ldots,r)=\bar{\delta}+\bar{d}(\bar{y}+r). \quad (19)$$

Let us fix the risk premium $\bar{\delta}$ together with the risk-free rate r_f and analyse the equilibria for $\bar{d}$ varying. Cumbersome computations result in proposition 3.1 and figure 1 of Chiarella and He (2001). One can immediately reproduce these results looking at the EMC plot. Indeed, the linear function (19) in this case passes through the point $M=(-\bar{y},\bar{\delta})$, which does not depend on $\bar{d}$. The slope of this function is $\bar{d}$ and so the change of the extrapolation parameter translates into rotation of the line around point M. Three typical situations are presented in panel (b) of figure 5. The horizontal investment function corresponds to $\bar{d}=0$, i.e. to the fundamentalist type of behaviour. Analogously, any trend-follower possesses an increasing investment function, while the chartist's function is decreasing. The rotation argument immediately explains panel (a) of figure 5 which is essentially figure 1 of Chiarella and He (2001). The meaning of points d_F, d_L and d_U becomes clear. The former point corresponds to the investment function of contrarians passing through point F of the EMC, while the two latter points correspond to the tangency between the investment function and the EMC.

It is also useful to represent the family of functions (18) on the stratification diagram. Since function (19) in representation (13) has coefficients $A^{\mathrm{CH}}=\bar{\delta}+\bar{d}r_f-1$ and $B^{\mathrm{CH}}=\bar{d}$, all couples (A^{CH}, B^{CH}) lie on the straight line with positive slope marked by the corresponding arrow in figure 4. Movement along the line corresponds to the change in the extrapolation parameter $\bar{d}$. Notice that the line intersects regions I, II, VII, V and (ultimately) IV, which is consistent with figure 5.

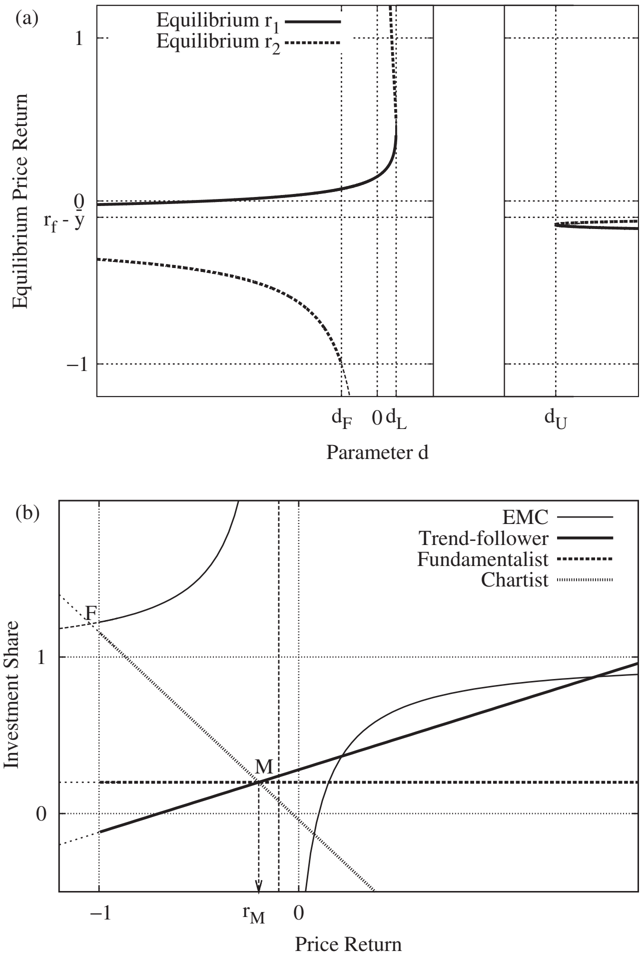

Figure 5. Example of linear investment functions: the case of homogeneous agents in the model of Chiarella and He (2001). Panel (a): equilibria as a function of extrapolation parameter $\bar{d}$ (see figure 1 of the original paper). Panel (b): investment functions for the contrarian, fundamentalist and trend-follower.

3.3. *Equilibria for the EU maximizers*

In the previous section the EMC plot was effectively used to study the effects of change of different parameters of the demand functions. The real advantage of our geometric approach, however, can be seen in the situations when the demand function is not known explicitly. One such case arises when agents perceive any non-trivial distribution of the future wealth and maximize the EU with power function (6).

The way to resolve the problem is to derive some approximation of the solution. There are some issues with this approach. First, different approximations are possible. For instance, Chiarella and He (2001) use the continuous-time approximation and obtain the mean–variance share (5) with the risk-aversion coefficient γ. Campbell and Viceira (2002) derive another continuous-time approximation (formula (2.25) on p. 29) which differs from (5) on the constant term. Second, even if an approximation is precise in the limit when the time unit converges to zero, the error of the approximation for the actual time scale can be large. Usually, no estimation of the error incurred due to the approximation is provided, and so the reliability of such an approach is under question. Third, it may happen that the additional specific assumptions imposed on the return distribution (necessary to derive the approximation) are in contradiction with the realized dynamics. In this case the approximation is no longer justified.

To overcome the problems with approximation let us, first, derive some properties of the actual solution. The EU maximization problem with power utility (6) leads to an investment function which depends on the risk-aversion coefficient γ and on the agent's belief about the distribution of future excess return $z_{t+1}=r_{t+1}+y_{t+1}-r_f$. Let us denote this perceived distribution as $g(z)$, the expected value of the excess return as $\bar{z}$, and the corresponding investment function as $f^{\mathrm{EP}}(\gamma, g(z))$. Then the following applies.

Proposition 3.5: *Let f^{EP}_γ be the partial derivative of the investment function f^{EP} with respect to the risk-aversion coefficient γ*

$$\textit{if } \bar{z} \gtreqless 0, \quad \textit{then } f^{EP} \gtreqless 0 \quad \textit{and} \quad f^{EP}_\gamma \lesseqgtr 0.$$

Proof: See Appendix B. □

This proposition together with the general results of section 3.1 allow us to discuss some of the equilibrium properties of the market with the EU maximizers even without complete knowledge of their investment functions. For instance, it is immediately clear that such a market cannot be in equilibrium with $r^* + \bar{y} < r_f$. Such equilibria lie in the left branch of the EMC where the agents have positive investment shares. Therefore, they have to expect positive excess return, which is inconsistent with the realized return.

Furthermore, proposition 3.5 implies that, independently of the agent's perceived distribution of the return, an increase in risk aversion will result in a downward movement of those parts of the investment functions which lie above the horizontal axes and in an upward movement of those parts which are below the axes. It opens the way for the comparative statics. Assume that the market populated by the EU maximizers is in equilibrium. As shown above, these equilibria lie in the right branch of the EMC. If the agents expect positive (negative) excess return $\bar{z}$, then according to proposition 3.5 they have to have a positive (negative) investment share and an increase in their risk aversion leads to some shift of the investment function down (up). Looking at the EMC, one concludes that the equilibrium return r^* will decrease (increase).

4. Stability analysis

The EMC can help with the comparative statics analysis. However, such an analysis is not very informative if the equilibria under consideration are not stable. In this section the stability analysis of the equilibria will be performed. It is organized much in the same way as section 3; after the presentation of the general results the applications to the special cases are discussed.

4.1. Stability of the general system

The local stability conditions are derived from the analysis of the roots of the characteristic polynomial associated with the Jacobian of the corresponding system computed at equilibrium. The complete stability analysis for all equilibria of (8) was performed by Anufriev and Bottazzi (2006). Here we only reproduce some relevant parts of that analysis and proceed in two steps.

4.1.1. Evolutionary selection of agents. The following statement provides the first set of stability conditions.

Proposition 4.1: *Let* $\mathbf{x}^*$ *be an EP equilibrium of system* (8), *found in proposition* 3.1(i), *where the first k agents survive. The fixed point* $\mathbf{x}^*$ *is* (*locally*) *stable if the following two conditions are met*:

(i) *the equilibrium investment shares of the non-surviving agents satisfy the relations*

$$-2 - r_f - r^* < x^*_n(r^* + \bar{y} - r_f) < r^* - r_f, \quad \textit{for } k < n \le N; \tag{20}$$

(ii) *after the elimination of the non-surviving agents, the same equilibrium is locally stable for the corresponding reduced system.*

Proof: See Appendix F of Anufriev and Bottazzi (2006). □

To understand this result notice that condition (ii) is independent of the behaviour of the non-survivors. It states that the stable equilibria are 'self-consistent', i.e. they remain stable after all non-surviving agents are removed from the economy. We investigate this condition below. Now turn to condition (i) and specifically to the rightmost inequality in (20). Remember that the survivor's investment share $x^*_{1\diamond k} = (r^* - r_f)/(r^* + \bar{y} - r_f)$. Thus, in the equilibria on the right branch of the EMC (with positive equity premium) this inequality reduces to $x^*_n < x^*_{1\diamond k}$, while on the left branch of the EMC (with negative equity premium) it becomes $x^*_n > x^*_{1\diamond k}$. In other words, the survivors must be the most aggressive investors in equilibria where the risky asset is the more attractive of the two, otherwise they have to be the least aggressive investors. This result is a direct consequence of the replicator dynamics governing the evolution of the wealth shares (see the last equation in (8)) and provides the condition for evolutionary stability against invasion of the market by another agent. An incumbent would start to lose her wealth if an infinitesimally small initial wealth is assigned to an agent with higher share to invest in the most attractive asset of the two.

In panel (a) of figure 6 we report in grey those regions where inequalities (20) hold. For the same market as in panel (b) of figure 1 equilibrium S_2 turns out to be unstable, since the investment function of the non-surviving agents there has greater value and does not belong to the grey area. Analogously, U_1 is unstable since agent II is more aggressive in this equilibrium. On the other hand, in U_2 and in all the equilibria in S_1, condition (i) of proposition 4.1 is satisfied.

4.1.2. Stability of equilibria without non-survivors. Consider now the second condition of proposition 4.1. When all the non-survivors are eliminated, the reduced system still has the same form (8). When is the equilibrium of this system stable? Let us first discuss the simplest case and then generalize.

Case of one survivor with L = 1. The wealth dynamics is irrelevant in the reduced system with a single survivor. If the memory span $L = 1$, the system becomes two dimensional, and one obtains the following.

Proposition 4.2: *The fixed point of system* (8) *with a single investment function* $x_{t+1}=f(r_t)$ *is* (*locally*) *asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)}\frac{1+r_f}{r^*-r_f}<1,\quad \frac{f'(r^*)}{l'(r^*)}<1,\quad \frac{f'(r^*)}{l'(r^*)}\frac{2+r^*+r_f}{r^*-r_f}>-1, \quad (21)$$

where $f'(r^*)$ *and* $l'(r^*)$ *are the first derivative of the investment function* $f(r)$ *and of the EMC* $l(r)$ *computed in equilibrium*, *respectively*.

The equilibrium is unstable if at least one of the inequalities in (21) *holds with the opposite* (*strict*) *sign. The stability is lost through a* Neimark–Sacker, fold *or* flip *bifurcation if the first*, *the second or the third inequality*, *respectively*, *is violated*.

Proof: See appendix C. □

The region where conditions (21) are satisfied is shown in panel (b) of figure 6 in coordinates r^* and $f'(r^*)/l'(r^*)$. The second coordinate is the relative slope of the investment function at equilibrium with respect to the slope of the EMC. If the slope of f at the equilibrium increases, the system tends to lose its stability.

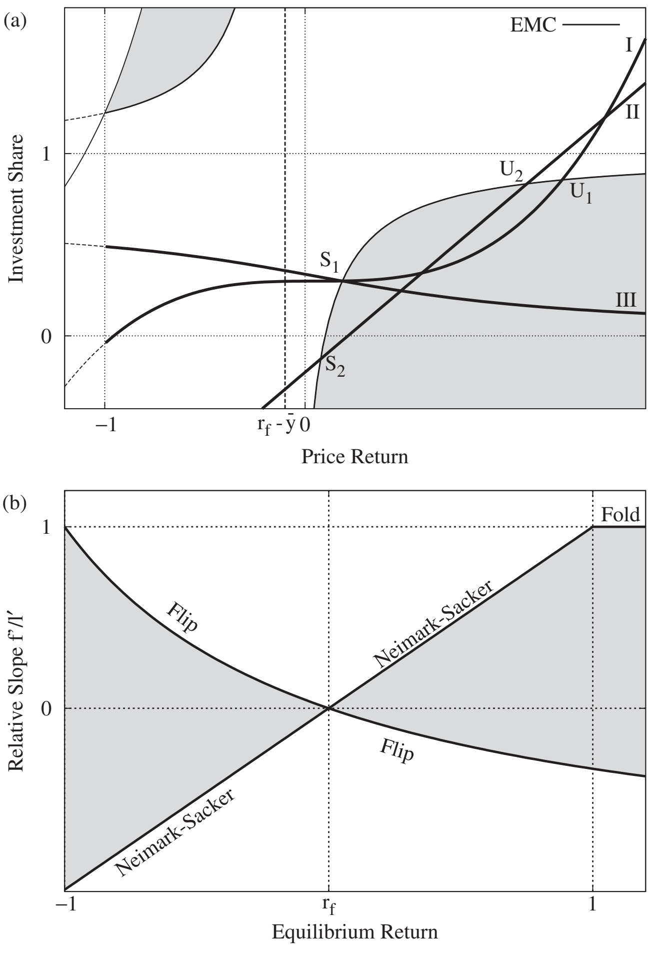

Figure 6. Stability for the multi-agent system. Panel (a): the region where investment shares of the non-surviving agents satisfy the stability conditions (20) is shown in grey. Panel (b): equilibrium stability region (grey) and the bifurcation types for the single-agent case with $L = 1$ in coordinates r^* and $f'(r^*)/l'(r^*)$

In particular, in the stable equilibrium the slope of the investment function is smaller than the slope of the EMC. Once again, let us consider the market in panel (a) of figure 6 and suppose for the moment that these are the functions of the agents with memory span 1. We immediately see that equilibria U_1 and U_2 are unstable, due to the violation of the second inequality in (21).

General case. In general, the stability depends on the behaviour of the individual investment functions in an infinitesimal neighbourhood of the equilibrium x^*. In contrast to the single-survivor case the investment functions of all survivors are important for stability, and in contrast to the case with $L=1$ the derivatives with respect to different variables matter.

Introduce the *stability polynomial* of the investment function f in x^* as

$$P_f(\mu)=\frac{\partial f}{\partial r_{t-1}}\mu^{L-1}+\frac{\partial f}{\partial r_{t-2}}\mu^{L-2}+\cdots+\frac{\partial f}{\partial r_{t-L+1}}\mu+\frac{\partial f}{\partial r_{t-L}}, \quad (22)$$

where all the derivatives of f are computed in the point $(r^*,\ldots,r^*)$. The general stability conditions can be formulated as follows.

Proposition 4.3: *Let* $\mathbf{x}^*$ *be a fixed point of system* (8), *where all* k *agents survive. Let* $P_{f_n}(\mu)$ *be the stability polynomial of the investment function* f_n. *The equilibrium* $\mathbf{x}^*$ *is* (*locally*) *stable if all the roots of the polynomial*

$$Q_{1\diamond k}(\mu)=\mu^{L+1}-\frac{(1+r^*)\mu-(1+r_f)}{(r^*-r_f)l'(r^*)}\sum_{n=1}^{k}\varphi_n^* P_{f_n}(\mu) \quad (23)$$

lie inside the unit circle.

Proof: See Appendix F of Anufriev and Bottazzi (2006). □

The analysis of the roots of $Q_{1\diamond k}(\mu)$ can be used to reveal the role of the different parameters in stabilizing or destabilizing a given equilibrium. Such an analysis is quite complicated and almost no general conclusions can be obtained. Notice, however, that if the survivors' investment functions are horizontal in the equilibrium, all the roots of polynomial $Q_{1\diamond k}$ are zeros and x^* is stable. Since the roots of a polynomial are continuous functions of its coefficients, we conclude that equilibria with flat enough investment functions are stable, similar to the case $L=1$. Furthermore, since the stability polynomials of the investment functions are weighted in (23), the equilibria can be stabilized if the survivors with the steep investment functions have small enough relative wealth.

4.1.3. Optimal selection in the equilibrium.

Before applying the stability results to the special cases, let us have another look at proposition 4.1. The total wealth in the economy asymptotically coincides with the wealth accumulated by the survivors. Therefore, the inequality on the right-hand side of (20) can also be interpreted as follows. The economy never ends up in a situation where its growth rate is lower than it would be if the survivors were

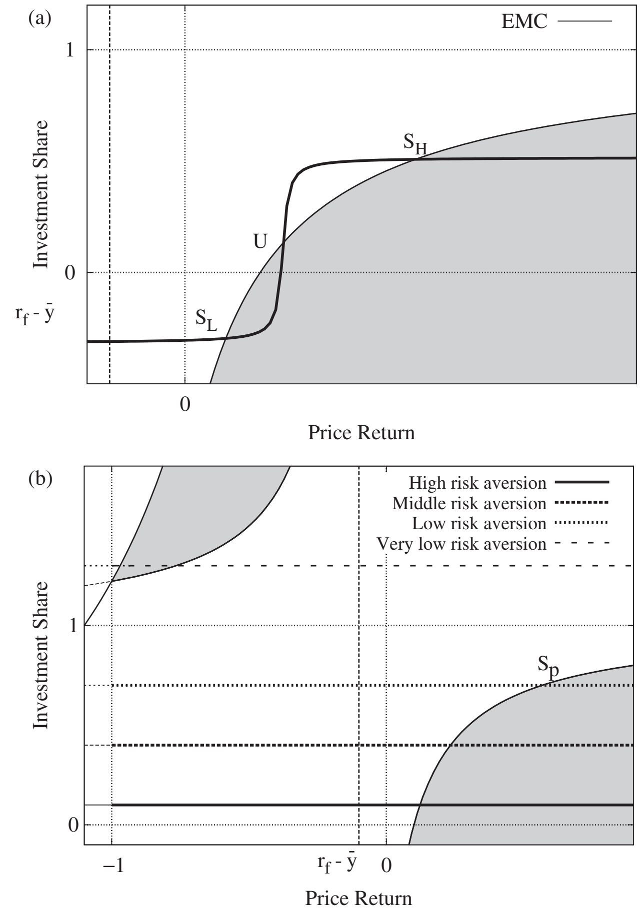

Figure 7. Panel (a): in the market with a single agent, multiple stable equilibria can be generated by a nonlinear investment function. Panel (b): equilibria in the market with the EU maximizing fundamentalists.

substituted by some other agents. This result can be called an *optimal selection principle* since it suggests that the market endogenously selects the best aggregate outcome.

It is important to keep in mind two limitations of this principle. First, condition (ii) of proposition 4.1 indicates that the principle does not apply to the whole set of equilibria. Namely, the reduced system should satisfy the conditions of proposition 4.3. For instance, the market in panel (a) of figure 6 will never end up in U_2 even if these are the equilibria with the highest returns. Second, there are single investment functions generating multiple stable equilibria, as the example in panel (a) of figure 7 demonstrates. Equilibria S_L and S_H are stable, as the investment function is (almost) horizontal in these two points. Thus, the optimal selection principle has only a local character: the economy does not necessarily converge to the stable equilibrium with the highest possible return.

4.2. *Stability for a single linear investment function*

In section 3.2 we characterized possible equilibria in the market where a single agent has investment functions with linear symmetrization. Can we tell something about their stability? One problem here is that the assumption of linearity of the symmetrization does not provide any information about the L partial derivatives of the function f in equilibrium, which appear in the stability polynomial (22). However, if $L=1$, then proposition 4.2 gives stability conditions explicitly. Therefore, we strengthen here the assumption about the linear form (13) for the 'symmetrization' of the investment function and assume that the investment function *itself* is linear:

$$f(r) = (A+1) + B(r + \bar{y} - r_f). \tag{24}$$

A linear investment choice based on a naïve forecast of the future return represents a possible interpretation of such behaviour. From (21) we obtain the following conditions sufficient for stability:

$$\frac{B}{l'(r^*)}\frac{1+r_f}{r^*-r_f} < 1, \quad \frac{B}{l'(r^*)} < 1, \quad \frac{B}{l'(r^*)}\frac{2+r^*+r_f}{r^*-r_f} > -1, \tag{25}$$

where r^* is the equilibrium return. Corresponding values of the return were computed in proposition 3.4. Plugging them into (25), one can express stability conditions and bifurcation loci through parameters A and B. The resulting expressions are quite cumbersome, so we provide only their geometric illustration.

In figure 8 we consider the parametric space (A, B) and produce its stratification according to the validity of the stability conditions for both equilibria found in proposition 3.4. In each point of the space we compute the corresponding equilibrium (if it exists) and check whether each of the three inequalities (25) holds. In the grey regions the corresponding equilibrium exists, it is feasible and stable. Otherwise, the parameter couple belongs to the white region. Apart from the 'tangency' and 'feasibility' curves (thick lines) which we used to obtain figure 4, we show as dotted thick lines different *bifurcation loci*. They correspond to the points where one of the inequalities (25) changes its sign. For example, the convex parabola corresponds to those points where the first inequality changes its sign. In these points the system exhibits the Neimark–Sacker bifurcation. Analogously, the concave parabola in panel (a) and another concave parabola in panel (b) represent points of the flip bifurcations, where the third inequality (25) changes sign.

Figure 8 suggests that, even if two feasible equilibria can coexist for linear investment functions, at least one of them will be unstable. We prove this in the following.

Proposition 4.4: *There is at most one feasible stable equilibrium in the market with the single linear investment function* (24).

Proof: See appendix D. □

This proposition shows that the restriction of the analysis to the market populated by the agents with linear investment functions (in particular, those who derive their demand through the MV optimization) leads to the impossibility of having the phenomenon of multiple *stable* equilibria in the single-agent case. If nonlinear investment functions were allowed, many stable equilibria could co-exist. As a consequence of this limitation, the range of

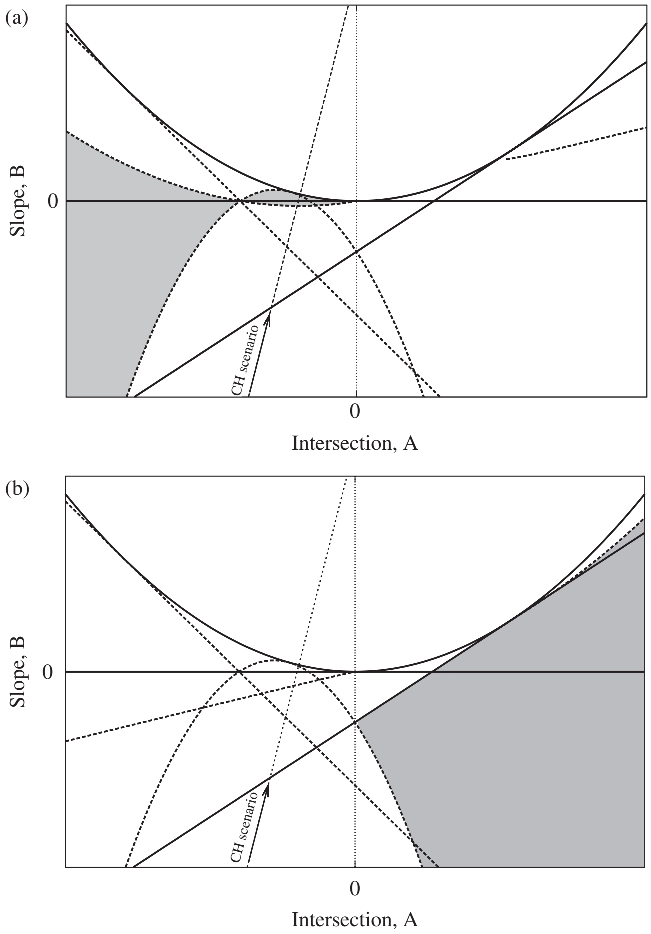

Figure 8. Stratification of the parameter space (A, B) according to the stability of the equilibria. Panel (a): stability of the first root r_1^*. Panel (b): stability of the second root r_2^*. The corresponding root is stable if the parameters belong to the grey area.

possible market dynamics can be oversimplified if only 'linear' behaviours are considered.

4.3. *Stability for the EU maximizers on the EMC*

In section 3.3 it was shown that even if the investment functions are not given explicitly, the EMC can help in the comparative statics analysis. The stability results, especially proposition 4.1, enrich this analysis even further. Let us consider the population of the fundamentalists, i.e. EU maximizers with homogeneous expectations which do not depend on the past returns. This assumption implies that the investment functions are horizontal. Depending on the sign of the expectations for the excess return, $r_t + y_t - r_f$, two cases are possible.

If expectations are positive, then the investment functions are positive (proposition 3.5). Now, from the general stability analysis it follows that if the investment shares of all the agents are less than 1, then the only stable equilibrium is generated by the most aggressive agent. Using once again proposition 3.5 we conclude that only the agent with the smallest risk aversion survives in the stable equilibrium S_p (see panel (b) of figure 7). Notice, however, that if such a market has an agent with so low risk aversion that she is willing to go short in the riskless asset, the situation without stable equilibrium can arise.

Analogously, if fundamentalists believe that the excess return will be negative, their investment functions lie below 0 and now the agent with the highest risk-aversion coefficient will survive in the stable equilibrium. Notice, however, that, in this equilibrium, the realized excess return is positive, which is inconsistent with the expectations.

5. Equilibria with multiple mean–variance investors

Let us start with a brief review. In section 2.2 it was shown that the mean–variance optimization leads to a demand function consistent with the constant relative risk-aversion framework. In section 3.2 the special property of the corresponding investment functions, namely linearity, was identified and the consequences for the location of equilibria were investigated. Finally, in section 4.2 it was proven that the market with such a single function cannot have multiple stable equilibria. The last step consists of the analysis of the market with *heterogeneous* MV optimizers. Such an analysis has been performed by Chiarella and He (2001), henceforth CH. In this section we reconsider it using the geometric tool of the EMC.

5.1. *Model of Chiarella and He: review of the results*

CH consider agents with investment function f^{CH} given in (18). All these agents have the same risk-aversion coefficient $\gamma = 1$. First, the model with homogeneous expectations is analysed. The risk premium $\bar{\delta}$ of the identical demand functions of agents is fixed and the extrapolation parameter $\bar{d}$ is changing, so that the situations of fundamental, trend-following or contrarian behaviour as described in section 3.2 are possible. Stability analysis is performed for the case $\bar{d} = 0$, when the unique equilibrium is asymptotically stable, and for the case when $\bar{d} \neq 0$ and $L = 1$, when sufficient conditions for stability are derived (corollary 3.3). For larger memory span L the numerical approach is exploited, which shows that the stability can be brought to the system through increase of the memory span. The qualitative aspects of the equilibrium and stability analysis of the single-agent case are summarized in figure 1 of that paper.

Second, the market with two investors is analysed and four different scenarios are considered. In the first scenario there are two fundamentalists with different risk premia. The equilibrium analysis shows that there are two equilibria in such a market (proposition 4.2), but only one of them is stable (corollary 4.3). It leads to an 'optimal selection principle' for this scenario, which states that the investor with the higher risk premium will survive.

The second scenario corresponds to the market with one fundamentalist and one contrarian. There exist three steady states for such a market, but the price return is positive in only two of them (proposition 4.4). The fundamentalist dominates the market in one of these two steady states and the contrarian dominates the market in another one. The stability analysis can be performed analytically only for the former steady state (corollary 4.5). As a result of numerical analysis of the stability of the latter steady state, CH conclude that the long-run return dynamics depend on the relative levels of the returns in these two steady states and follow a similar optimal selection principle. Namely, the steady state is stable if it generates the highest return.

In the third example of a heterogeneous market a fundamentalist meets a trend-follower. Such a market has one equilibrium where the fundamentalist survives. It also can have zero, one or two equilibria with surviving trend-follower (proposition 4.6). Similar to the previous example, the stability conditions for the latter equilibria are obtained through numerical investigation. It is found that, for small extrapolation rates (i.e. for a relatively small value of $\bar{d}$ of the trend-follower), there exist two equilibria where the trend-follower survives. The highest return is generated in one of these equilibria, which is, however, unstable. Between the two remaining equilibria "the stability switching follows a (quasi-)optimal selection principle", depending on where the return is higher.

Finally, in their last example, CH consider the market with two chartists. In this case there exist multiple steady states. If traders extrapolate strongly, one of the steady states is stable. For weak extrapolators, "the stability of the system follows the (quasi-)optimal selection principle—the steady state having relatively higher return tends to dominate the market in the long run".

To summarize, Chiarella and He have found a *quasi-optimal selection principle* which allows the prediction of the long-run market dynamics in the case when there are multiple equilibria. There is an important difference between this and the optimal selection principle which we formulated in section 4.1.3.

The principle in CH has a *global* character. When the ecology of the traders is fixed, it can be applied to the market, so that a unique possible outcome is predicted. Our optimal selection principle has a *local* character, instead. For a given traders' ecology there can be different possibilities of the market long-run behaviour, i.e. multiple equilibria. The final outcome depends on the initial conditions and, in the stochastic case, on the yield dynamics, and cannot be predicted *a priori*. However, independently of the realized equilibria, the survivors will be chosen in an 'optimal' way: to allow the highest possible growth rate of the economy in this point. In some sense, our principle selects among investment functions, while the principle in CH chooses among equilibria.

5.2. Model of Chiarella and He: geometric approach

We have already seen that analytic results of equilibrium analysis of the CH model for the single-agent case become clearer if one uses geometric tools. Stability analysis for the case $L=1$ can also be illustrated in the stratification diagram in figure 8. In particular, any horizontal (fundamental) investment function is stable, and such an equilibrium remains stable for $\bar{d}$ close to 0. Moreover, the equilibrium r_1^* is stable for very small negative $\bar{d}$, while the equilibrium r_2^* is stable for very large positive $\bar{d}$.

Now we turn to the two-agent case and illustrate four scenarios in Figure 9 with the EMC plot. Consider, first, the case of two fundamentalists with different risk premia $\bar{\delta}_1 > \bar{\delta}_2$ (panel (a), first row). These traders have horizontal investment functions passing through the points $M_1 = (-\bar{y}, \bar{\delta}_1)$ and $M_2 = (-\bar{y}, \bar{\delta}_2)$. From the assumption on the risk premia it follows that M_1 is above M_2. There are two equilibria in such a market: S and U. Each of these equilibria would be stable if the corresponding agent were to operate alone. When the two agents operate together, then equilibrium S with the highest risk premium is stable, while U is unstable. Notice that this result can immediately be generalized for the case of an arbitrary number of fundamentalists.

Let us now suppose that a fundamentalist with risk premium $\bar{\delta}_1$ encounters in the market a contrarian with risk premium $\bar{\delta}_2$, so that horizontal and decreasing investment functions are competing. CH distinguish between two cases depending on which of these risk premia is higher. Geometrically, it corresponds to the location of points M_1 and M_2. We start with the case in which $\bar{\delta}_1 \geq \bar{\delta}_2$, i.e. when point M_1 is above M_2 (panel (b), first row). With respect to the previous case we have made a rotation of the lower investment function around point M_2. It is obvious that the equilibrium S_f is always stable in this case, while the equilibrium S_c cannot be stable. Thus, the left plot in figure 3 of CH illustrating the qualitative features of this situation is obtained.† In the second case, when $\bar{\delta}_1 < \bar{\delta}_2$, there are different possibilities. If the contrarian extrapolates not very strongly, so that an absolute value of $\bar{\delta}_2$ is small enough (panel (c)), then S_f is, certainly, an unstable equilibrium. Therefore, S_c remains to be the only candidate for the stable equilibrium in this market. It will be stable only when it is stable in the single-agent case, which happens for relatively small $\bar{d}_2$ (see panel (a) of figure 8). Otherwise, there is no stable equilibria in the market. If, on the other hand, the contrarian extrapolates strongly (panel (d)), then S_f is the only stable equilibrium. Comparing this analysis with the second graph in figure 3 of CH, we can see that the

†All plots in Chiarella and He (2001) which we mention here and below are just sketches obtained from the mixture of the analytic and numerical analysis. The advantage of our approach is that these qualitative sketches can be obtained from the EMC plot. Thus, on the one hand, they all become justified on an analytic basis. On the other hand, they also become clearer and, thus, can be easily generalized for the situations of three and more agents, and also corrected. For example, notice that, in this case, the return in equilibrium S_c *does not* approach the return in equilibrium S_f when $\bar{d}_2 \to 0$.

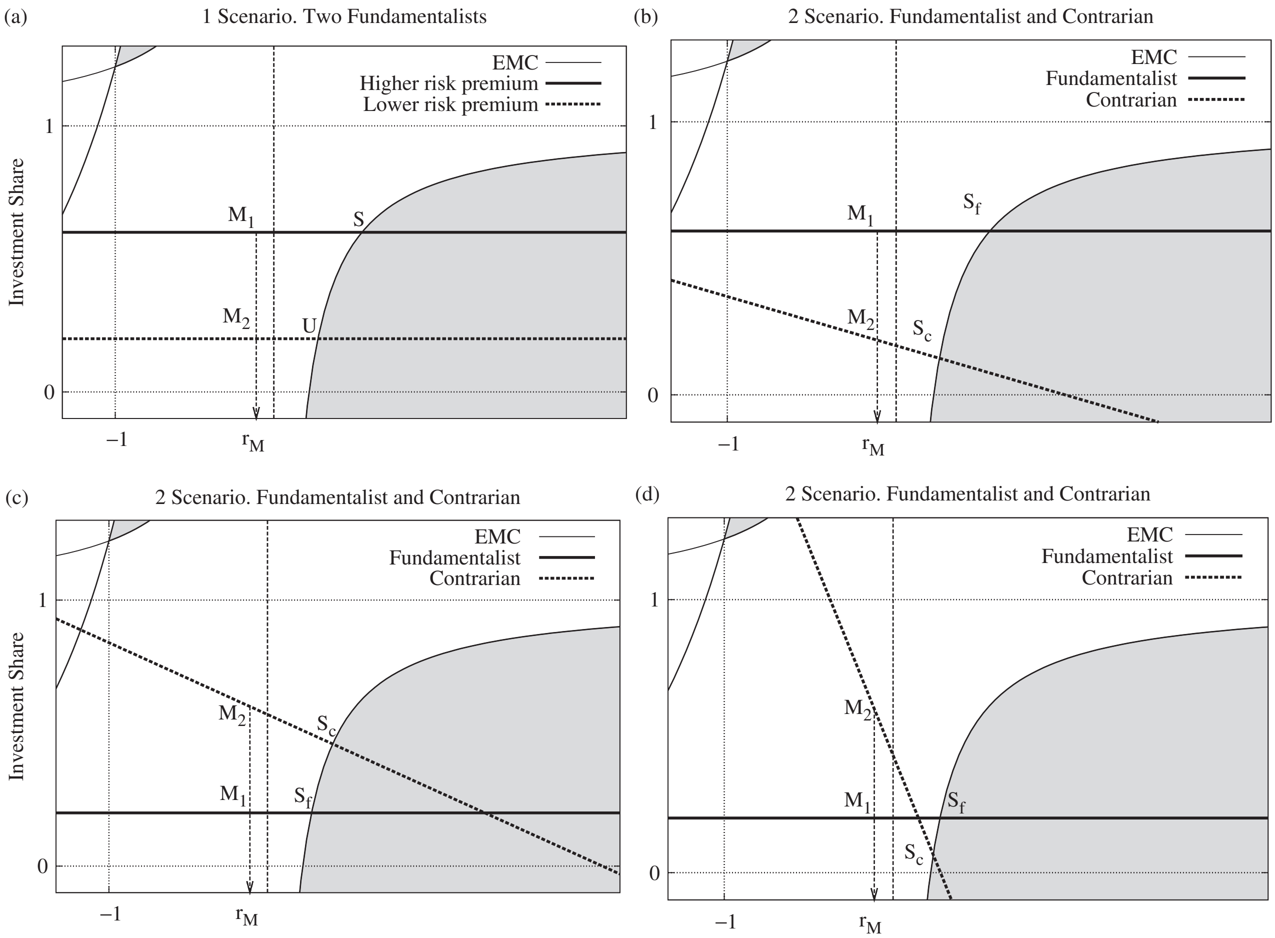

Figure 9. Equilibria in the model of Chiarella and He with two agents. See text for explanation.

situation of the possible absence of any stable equilibrium in the market has been overlooked.

In the third example we consider the case when the fundamentalist with the risk premium $\bar{\delta}_1$ competes with the trend-follower with the risk premium $\bar{\delta}_2$. In this example, we again distinguish between two cases depending on which of the risk premia is greater. Let us, first, assume that $\bar{\delta}_1 \geq \bar{\delta}_2$. There are two possibilities. If the trend-follower extrapolates not too strongly, the equilibrium S_t is not stable (panel (e)). The equilibrium S_f is stable in this case. If the trend-follower extrapolates more strongly, her investment function rotates and the equilibrium S_f loses its stability. S_t remains the only candidate for the stable equilibrium. If it exists and is stable in the market with the trend-follower alone, then it is also stable in the two-agent situations (panel (f)). Otherwise, there are no stable equilibria in the market with two agents. Such will be the case for $d_U > \bar{d}_2 > d_L$, since, in this situation, there is no equilibrium in the market with a surviving trend-follower. But it also happens for some $\bar{d}_2$ lower than d_L. Finally, for very strong extrapolation, when $\bar{d}_2 > d_U$, the market may have a stable equilibrium, if it exists, for the trend-follower. In the case when $\bar{\delta}_1 < \bar{\delta}_2$ (panel (g)) it is obvious that equilibrium S_f cannot be stable, therefore the market will have a stable equilibrium S_t whenever it is stable for the trend-follower, that is for small enough $\bar{d}_2$.

Finally, in panel (h) of figure 9 we consider an example when two technical traders coexist in the market. We draw the situation when both of them are trend-followers and have the same risk premium, so that their investment functions pass through the same point M. It is clear that the agent with the lowest extrapolation rate will generate the equilibrium S_2, which will always be unstable. Instead, equilibrium S_1 generated by the second agent will be stable if and only if it is stable in the single-agent market. Comparing it with panel (b) of figure 5 of CH we notice that, with further increase of d_2, the stable equilibrium (with growing return) becomes unstable and, eventually, disappears, so that, for higher extrapolation rates, the market does not have any equilibrium.

6. Conclusion

In this paper we have applied the general model of Anufriev and Bottazzi (2006) to a special class of agents' behaviour. For the application we have chosen the most common class in economics, namely the class of optimizing behaviour, and demonstrated that the model has implications for a very large subset of this class.

The generality of the Anufriev and Bottazzi framework together with the geometric representation of their results allowed us to overcome well-known technical difficulties in

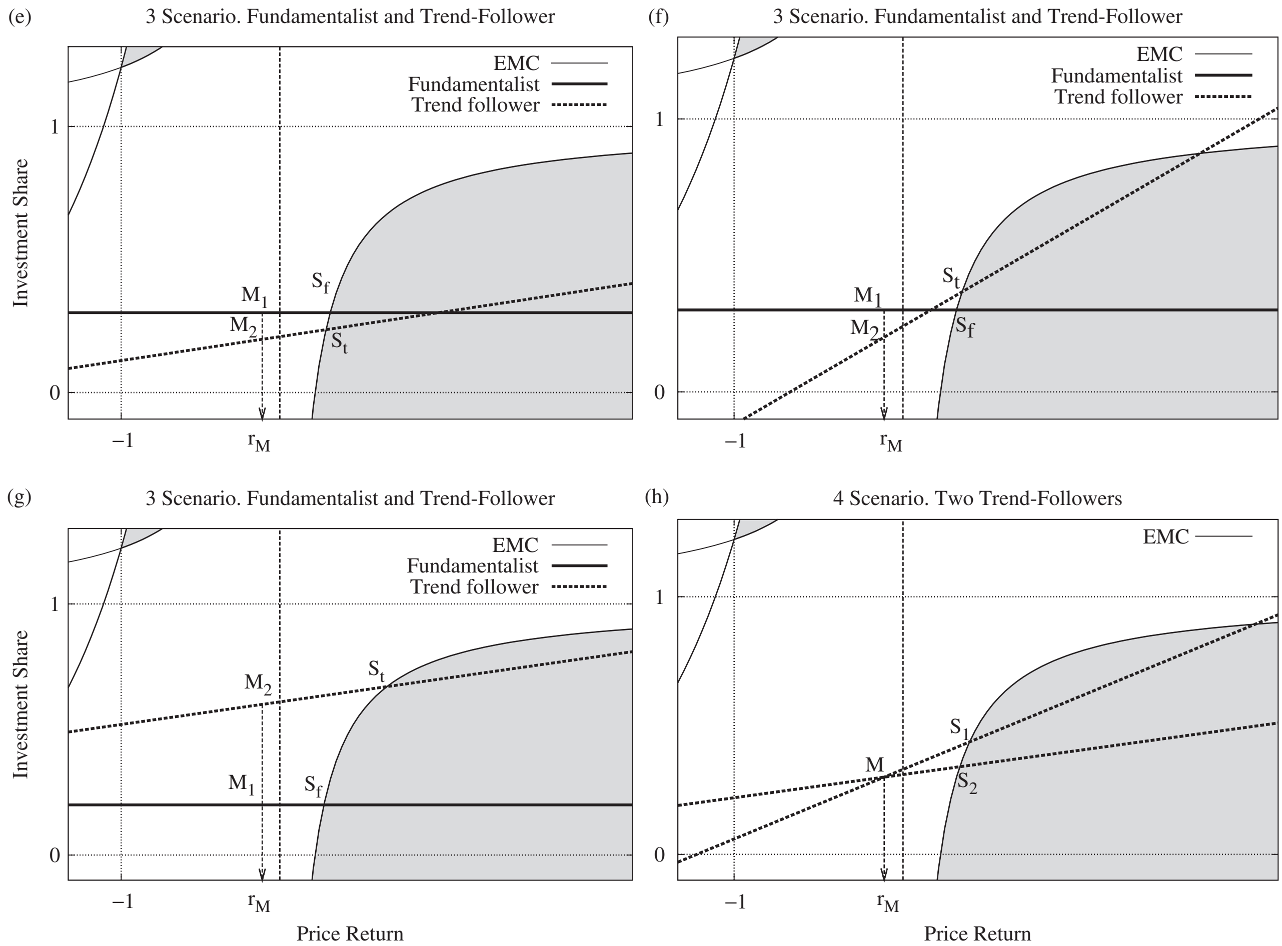

Figure 9. Continued.

the expected utility maximization setting. We have shown, for instance, that investment functions derived in this setting, which are only implicitly defined, shift downward with risk aversion. This immediately implies, given the geometric nature of the locus of all possible equilibria (the Equilibrium Market Curve), that the price return will decrease when the risk-aversion coefficient of the agents increases. This result is not new in the economics literature: if the agents are willing to take a smaller amount of risk, they will also obtain a smaller return. What is new, however, is that we have rigorously obtained this result from the framework with *endogenous* price setting.

We have also analysed the setting where the agents have mean–variance demand. In this case we have demonstrated that the qualitative results concerning market dynamics can be obtained using the EMC plot. As an application, we have shown that the analytic model with heterogeneous agents presented by Chiarella and He (2001) can be easily understood and generalized in many directions. Namely, the analysis can be extended for an arbitrarily large number of agents with arbitrary risk aversion and expectation rules. Probably, the easiest way to illustrate the advantages of the general approach is to have a look at the stratification diagrams in figures 4 and 8, drawn for a special, 'linear' case of the agent's behaviour. Even in this particular case, the scope of the model of Chiarella and He is represented by a one-dimensional straight line. Moreover, only a small interval of this line is analysed in that model, since the risk premium is assumed to be bound inside the interval $(0, 1)$.

In our view, the most interesting implication of this paper is that some features of the long-run market dynamics, such as multiple equilibria, cannot occur in a market with this specific population ecology. The global, quasi-optimal selection principle of Chiarella and He may hold when all demand functions are derived from mean–variance optimization, but it does not hold in general. In this respect, it seems promising, for further research, to apply the general framework from Anufriev and Bottazzi (2006) to other, non-rational, types of behaviour, e.g. those advocated by prospect theory or the behaviours based on threshold levels.

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Appendix A: Proof of proposition 3.4

In the case $B=0$, condition $l(r^*)=f(r^*,\dots,r^*)$ implies that $A+1=l(r)$, which is a linear equation with respect to r. We obtain (14) as soon as $A\neq 0$. If, instead, $B\neq 0$, then from the definition of the EMC we obtain the following quadratic equation with respect to $r+\bar{y}-r_f$:

$$B(r+\bar{y}-r_f)^2 + A(r+\bar{y}-r_f)+\bar{y}=0. \tag{A1}$$

The discriminant of this equation $D=A^2-4B\bar{y}$. Solving (A1) in the case when $D>0$, one obtains (15).

Appendix B: Proof of proposition 3.5

Let us introduce the following function:

$$h(x_t,\gamma)=\int z\,(1+x_t z)^{-\gamma} g(z)\mathrm{d}z,$$

where $g(z)$ is the perceived distribution of the next period excess return z. This distribution, in general, depends on the return history. The value of the investment function f^{EP} or, in other words, the investment share x^* of the agent who solves the EU maximization problem with power utility function (6), is the solution of the first-order condition (f.o.c.) $h(x_t,\gamma)=0$.

Let us first assume that $x^*>0$. Then, for both positive and negative z we have $z>z(1+x^* z)^{-\gamma}$. Multiplying both parts of this inequality on the function g, integrating with respect to z, and applying the f.o.c., we obtain $\bar{z}>0$. Analogously, if $x^*<0$, then $z<z\,(1+x^*\,z)^{-\gamma}$ for any $z\neq 0$, and, therefore, $\bar{z}<0$. Finally, when $x^*=0$ the f.o.c. implies that $\bar{z}=0$. This proves the first part of the statement.

The f.o.c.actually defines x^* as an implicit function of the risk-aversion coefficient γ. Applying the implicit function theorem, we obtain

$$f^{\mathrm{EP}}_{\gamma}=-\frac{1}{\gamma}\frac{\int z\log(1+x^*z)(1+x^*z)^{-\gamma}g(z)\mathrm{d}z}{\int z^2(1+x^*z)^{-\gamma-1}g(z)\mathrm{d}z}.$$

The denominator of the last expression is always positive, while the numerator is positive when $x^*>0$ and negative otherwise. This proves the second part of the statement.

Appendix C: Proof of proposition 4.2

We are dealing with a special case in which the system (8) has the form

$$x_{t+1}=f(r_t),$$
$$r_{t+1}=r_f+\frac{(1+r_f)(x_{t+1}-x_t)+\bar{y}x_t x_{t+1}}{x_t(1-x_{t+1})}$$

The Jacobian matrix J of this system at a fixed point reads

$$J = \left\| \begin{matrix} 0 & f' \\ -(1+r_f)/(x^*(1-x^*)) & (1+r^*)f'/(x^*(1-x^*)) \end{matrix} \right\|.$$

It is well known that the system is asymptotically stable if the following three conditions are satisfied: $d<1$, $t<1+d$ and $t>-1-d$, where t and d are the trace and determinant of the matrix J, respectively. Inequalities (21) are obtained by direct substitution taking into account that $l'(r^*) = x^*(1-x^*)/(r^*-r_f)$.

Appendix D: Proof of proposition 4.4

The constant investment function has either one or zero equilibria. For the increasing function, consider the second inequality in (25). Substitution of the EMC's slope in equilibrium leads to

$$B(\bar{y}+r^*-r_f)^2 - \bar{y} < 0 \Leftrightarrow -A(\bar{y}+r^*-r_f) - 2\bar{y} < 0,$$

where we have used relation (A1). Plugging in the corresponding equilibrium values from (15), one can simplify the resulting inequality using $B>0$ and obtain

$$\sqrt{A^2-4B\bar{y}} + A < 0 \quad \text{in } r_1^* \quad \text{and}$$
$$\sqrt{A^2-4B\bar{y}} - A < 0 \quad \text{in } r_2^*.$$

When $A>0$, the left inequality is violated and therefore r_1^* is unstable. Otherwise, r_2^* is unstable.

Consider now the case of decreasing investment function $B<0$. From (15) it follows that the equilibria are such that $r_2^* < r_f - \bar{y} < r_1^*$. If the equilibrium return is negative, the first inequality in (25) leads to

$$B(1+r_f)(\bar{y}+r^*-r_f)^2 - (r^*-r_f)\bar{y} > 0$$
$$\Leftrightarrow -A(1+r_f)(\bar{y}+r^*-r_f) - \bar{y}(1+r^*) > 0.$$

When $A \le 0$, it obviously always holds with the opposite sign in feasible r_2^*, i.e. r_2^* is always unstable in this case. Analogously, when r_1^* is negative it will be unstable when $A>0$.

The final case to consider is when both A and r_1^* are positive. The third inequality in (25) leads to

$$B(2+r^*+r_f)(\bar{y}+r^*-r_f)^2 + (r^*-r_f)\bar{y} > 0$$
$$\Leftrightarrow -A(2+r^*+r_f)(\bar{y}+r^*-r_f) - 2\bar{y}(1+r_f) > 0,$$

which is always violated. Thus, in all cases when two feasible equilibria exist, one of them is unstable.