Speculative Equilibria and Asymptotic Dominance in a Market with Adaptive CRRA Traders

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ABSTRACT

We consider a simple pure exchange economy with two assets, one riskless, yielding a constant return on investment, and one risky, paying a stochastic dividend. Trading takes place in discrete time and in each trading period the price of the risky asset is fixed by imposing market clearing condition on the sum of traders individual demand functions. Individual demands are expressed as fractions of traders wealth and depend on traders forecasts about future price movement. Under these assumptions we derive the stochastic dynamical system describing the evolution of price and wealth.

We study the cases in which one or two agents operate in the market, identifying the possible equilibria and discussing their stability conditions. The main novelty of our approach rests in the abstraction from the precise characterization of agents beliefs and preferences. In this respect our results generalize several previous contributions in the field. In particular, we show that, irrespectively of agents behavior, the system can only possess isolated generic equilibria where a single agent dominates the market and continuous manifolds of non-generic equilibria where heterogeneous agents hold finite shares of the aggregate wealth. Moreover, we show that all possible equilibria returns belong to a one dimensional "Equilibria Market Line". Finally we discuss the role of different parameters for the stability of equilibria and the selection principle governing market dynamics.

Keywords: Asset pricing model, CRRA Framework, Equilibria Market Line, Optimal selection principle

1. INTRODUCTION

This paper is devoted to the analysis of market equilibria that emerge from the interaction of different trading strategies in a speculative environment. We consider a simple pure exchange economy with two assets, a riskless security, yielding a constant return on investment, and a risky equity, paying a stochastic dividend. We consider the trading taking place in discrete time, define the riskless security as the numéraire of the economy and assume that in each trading period the price of the risky asset is fixed through the market clearing condition.

We derive the stochastic dynamical system which describes the evolution of price and wealth for the general case in which the economy is populated by a fixed number of heterogeneous traders. We explicitly identify the intertemporal constraints imposed on the investment choices of the agents by the requirement that the economy possesses, at each time step, a positive price.

We restrict the possible trading behavior of agents to consider only individual demand for the risky asset expressed as a fraction of the agent's present wealth. This class of behaviors contains, but is not limited to, strategies based on the expected utility maximization of a Constant Relative Risk Aversion utility function. This particular choice for the description of the individual behavior seems in line with the economic evidence[1] and is common to several relevant contributions to the literature.[2–5] With these contributions we share the assumption that the individual demand of agents depends on agents forecasts about future price movements and that these forecasts are exclusively based on past market performances. In other terms, using a finite set of statistical estimators, agents adapt their beliefs, and consequently their choices, to the recent history of the economy.

The novelty of our approach and a major point of departure with respect to the previous contributions, is the fact that we do not require agents investment choices to follow a specific analytical expression derived by the maximization of a uniquely defined CRRA utility function. While our results do perfectly apply to these

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particular cases, we consider a more general framework in which the investment choice of the agent can be any smooth function of his expectations about future prices. The importance of considering heterogenous, agent-specific *investment functions* rests in the possibility to describe different attitudes toward risk and, in general, different possible ways in which an agent can transform an available information set (public or private) into predictions about the future. We will show how, even if one leaves the explicit expression of these functions completely unspecified, the market dynamics can be characterized, at least locally, in terms of a few simple parameters.

Inside this general framework, we start with the analysis of the easiest situation in which a single trader operates in the market. We characterize all possible market equilibria and discuss the conditions for their stability. We show that, irrespectively of the agent's investment function, the price return at equilibrium always belongs to a one dimensional curve that we call *Equilibria Market Line* (EML). This curve allows for a simple geometric discussion of the system asymptotic behavior which straightforwardly generalizes the result obtained in Sect. 3 of Ref. 4.

We then move to the analysis of the economy with two distinct traders. Again with the help of the EML we discuss the existence of equilibria and their stability and, at the same time, compare the relative performances of different investment functions at equilibrium. We show that in the two traders economy only two types of equilibria are possible: generic equilibria, associated with isolated fixed points, where a single agent asymptotically possesses the entire wealth of the economy and non generic equilibria, associated with continuous manifolds of fixed points, where both agents possess a finite shares of the total wealth.

Using EML geometric interpretation of equilibria we are able to discuss the validity and limits of the "quasi-optimal selection principle" formulated in Ref. 4 for linear demand functions, when more general traders behaviors are taken into consideration. Our analysis reveals possible existence of multiple, isolated, locally stable equilibria and, consequently, the local nature of relative performances of different investment choices. This result can be interpreted as an "impossibility theorem" for the construction of a dominance order relation inside the space of trading strategies.

The rest of the paper is organized as follows. In Section 2 we describe our simple pure exchange economy, writing explicitly the traders inter-temporal budget constraint and laying down the equations (and constraints) governing the dynamics of the market. In Section 3 we present the case in which a single trader operates in the market. Section 4 contains the two agents case and a discussion of some implications of our main results. In Section 5 we summarize our findings and list some possible future directions of research.

2. MODEL STRUCTURE

We consider a simple pure exchange economy, populated by a fixed number N of traders, where trading activities take place in discrete time. The economy is composed by a risk-less security giving in each period a constant interest rate $r_f > 0$ and a risky asset paying a random dividend D_t at the end of each period t. The risk-less security is considered the numéraire of the economy and its price is fixed to 1. The price P_t of the risky asset is determined at each period, on the basis of its aggregate demand, through market-clearing condition. The dynamics proceeds as follows.

Let $W_{t,n}$ be the wealth of agent n at time t and $x_{t,n}$ the fraction of wealth this agent wants to invest in the risky asset. Suppose that, after the trading session at time $t-1$, agent n possessed $x_{t-1,n}\, W_{t-1,n}/P_{t-1}$ shares of risky asset and $(1-x_{t-1,n})\, W_{t-1,n}$ shares of risk-less asset. At this moment he received dividends payment D_{t-1} per each risky asset and payment of risk-less interest r_f on the wealth invested in the risk-less security. Therefore, at time t the wealth of agent n, for any notional price P, reads

$$W_{t,n}(P) = (1-x_{t-1,n})\, W_{t-1,n}\,(1+r_f) + \frac{x_{t-1,n}\, W_{t-1,n}}{P_{t-1}}\,(P + D_{t-1}) \tag{1}$$

and its individual demand for the risky asset becomes $x_{t,n}\, W_{t,n}(P)/P$. The actual price of the risky asset at time t is fixed at the level for which aggregate demand is equal to aggregate supply. Assuming a constant supply of risky asset, whose quantity can then be normalized to 1, the price P_t is defined as the solution to the equation

$$\sum_{n=1}^{N} x_{t,n}\, W_{t,n}(P_t) = P_t \quad . \tag{2}$$

At the end of period t the dividend D_t and the risk-free interest r_f are paid and the trading session at time $t+1$ can start.

The dynamics defined by Eqs. (1) and (2) describes an exogenously growing economy due to the continuous injections of new riskless assets, whose price remains, under the assumption of totally elastic supply, unchanged. To remove this exogenous expansion from the dynamics of the model we introduce rescaled variables

$$w_{t,n} = W_{t,n}/(1+r_f)^t \ , \quad p_t = P_t/(1+r_f)^t \ , \quad \text{and} \quad e_t = D_t/(P_t\,(1+r_f)) \ .$$

The market intertemporal relations written in terms of these new variables read

$$\begin{cases} p_t = \displaystyle\sum_{n=1}^{N} x_{t,n}\, w_{t,n} \\ w_{t,n} = w_{t-1,n} + w_{t-1,n}\, x_{t-1,n} \left(\dfrac{p_t}{p_{t-1}} - 1 + e_{t-1} \right) \quad \forall n \in \{1,\ldots,N\} \end{cases} \quad . \tag{3}$$

The market dynamics implies a simultaneous determination of the equilibrium price p_t and of the agents wealths $w_{t,n}$. Due to this simultaneity, the $N+1$ equations in (3) define the state of the system at time t only implicitly. Indeed, the N variables $w_{t,n}$ defined in the second equation appear on the right-hand side of the first, and, at the same time, the variable p_t defined in the first equation appears in the right-hand side of the second. For analytical purposes, one has to derive the explicit equations that govern the system dynamics.

Explicit Dynamics of Wealth and Return

The transformation of the implicit intertemporal relations (3) into an explicit map defining a meaningful economic dynamics, i.e. generating trajectories in which asset price remains positive, entails some restrictions on the possible market positions available to the agents. Before deriving these conditions and the explicit map, it is useful to introduce a notation which allows to present the dynamics in a more compact form.

Let a_n be an agent specific variable, dependent or independent from time t. We denote with $\langle a \rangle_t$ the *wealth weighted average* of this variable at time t on the population of agents, i.e.

$$\langle a \rangle_t = \frac{\sum_{n=1}^{N} a_n\, w_{t,n}}{w_t} \quad , \quad \text{where} \qquad w_t = \sum_{n=1}^{N} w_{t,n} \quad . \tag{4}$$

The next result gives the condition for which the dynamical system implicitly defined in Eq. (3) can be made explicit without violating the requirement of positiveness of prices

THEOREM 2.1. *From Eq. (3) it is possible to derive a map* $\mathbb{R}^{+N} \to \mathbb{R}^{+N}$ *that describes the evolution of traders wealth* $w_{t,n}$ $\forall n \in \{1,\ldots,N\}$ *with positive prices* $p_t \in \mathbb{R}^+$ $\forall t$ *provided that*

$$\Big(\langle x_t \rangle_t - \langle x_t\, x_{t+1} \rangle_t \Big) \Big(\langle x_{t+1} \rangle_t - (1-e_t) \langle x_t\, x_{t+1} \rangle_t \Big) > 0 \qquad \forall t \quad . \tag{5}$$

If this is the case, the price growth rate $r_{t+1} = p_{t+1}/p_t - 1$ *reads*

$$r_{t+1} = \frac{\langle x_{t+1} - x_t \rangle_t + e_t \, \langle x_t\, x_{t+1} \rangle_t}{\langle x_t\,(1 - x_{t+1}) \rangle_t} \tag{6}$$

and the evolution of wealth, described by the wealth growth rates $\rho_{t+1,n} = w_{t+1,n}/w_{t,n} - 1$*, is given by*

$$\rho_{t+1,n} = x_{t,n} \big(r_{t+1} + e_t \big) = x_{t,n} \frac{\langle x_{t+1} - x_t \rangle_t + e_t \, \langle x_t \rangle_t}{\langle x_t\,(1 - x_{t+1}) \rangle_t} \qquad \forall n \in 1,\ldots,N \quad . \tag{7}$$

Proof. Plugging the expression for $w_{t,n}$ from the second equation in (3) into the right hand-side of the first equation, assuming $p_{t-1} > 0$ and, consistently with Eq. (5), $p_{t-1} \neq \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}$, one obtains

$$p_t = \left(1 - \frac{1}{p_{t-1}} \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}\right)^{-1} \left(\sum x_{t,n}\, w_{t-1,n} + (e_{t-1} - 1) \sum x_{t,n}\, w_{t-1,n}\, x_{t-1,n}\right) =$$

$$= p_{t-1} \frac{\sum x_{t,n}\, w_{t-1,n} + (e_{t-1} - 1) \sum x_{t,n}\, w_{t-1,n}\, x_{t-1,n}}{\sum x_{t-1,n}\, w_{t-1,n} - \sum x_{t,n}\, x_{t-1,n}\, w_{t-1,n}} =$$

$$= p_{t-1} \frac{\big\langle x_t \big\rangle_{t-1} - \big\langle x_{t-1}\, x_t \big\rangle_{t-1} + e_{t-1} \big\langle x_{t-1}\, x_t \big\rangle_{t-1}}{\big\langle x_{t-1} \big\rangle_{t-1} - \big\langle x_{t-1}\, x_t \big\rangle_{t-1}} \quad ,$$

where the second equality comes from the first equation in (3) rewritten for time $t-1$. Condition (5) is immediately obtained imposing $p_t > 0$. Then the price return and wealth return for each agent n at time t can be derived straightforwardly. ∎

The explicit price dynamics can be immediately derived from Eq. (6), but the price will stay positive only as long as Eq. (5) is satisfied. The map defined by Eqs. (6) and (7) describes the evolution of the state variables $w_{t,n}$ and p_t over time, provided that the stochastic process $\{e_t\}$ is given and the set of investment shares $\{x_{t,n}\}$ is specified. The specification of agents investment decisions are presented in the next Section. Concerning the nature of the stochastic process we introduce the following

Assumption 2.2. *The dividend yields e_t are i.i.d. random variables with positive support and mean $\bar{e}$.*

Since this Assumption postulates the dividend yield to be independent from the price level, the system in Eqs. (6) and (7) is entirely defined in terms of price and wealth returns. The fixed points of this system correspond to asymptotic states of economic *steady growth**. Notice that Assumption 2.2 is common to several works in the literature†, so that a further reason to have this assumption introduced is to maintain comparability with previous investigations.

Agents Investment Functions

In the present work we are mainly concerned with the effect of speculative behaviors on the market aggregate performance. In order to eliminate the effects due to asymmetric evaluation and/or knowledge of the underlying fundamental process we model the agents investment choices as depending on the sole realized price returns, assuming, for any agent, a perfect knowledge of the fundamental $\{e_t\}$. Hence, the information set $\mathbb{I}_{t-1}$ commonly available to traders before the trading round at time t reduces to the set of past realized price returns

$$\mathbb{I}_{t-1} = \{r_{t-1}, r_{t-2}, \dots\} \quad .$$

We make the following

Assumption 2.3. *For each agent n there exists a deterministic* ***investment function*** *f_n which maps the present information set into its investment share:*

$$x_{t,n} = f_n(\mathbb{I}_{t-1}) \quad . \tag{8}$$

Function f_n in the right hand-side of Eq. (8) gives a complete description of the investment decision of agent n. Notice that according to Assumption 2.3, agents' investment decision is not affected by the past realizations of the fundamental e_t nor by agent-specific variables, like his own past investment choices or past investment choices of other traders.

*The dynamics defined by Eqs. (6) and (7) does not possess any interesting fixed point in terms of price and wealth levels, because of the constant expansion generated by the payment of positive dividends.

†In particular, it makes our economy identical to the one studied in Ref. 4.

In the majority of models discussed in the literature the investment choice described by Eq. (8) is obtained as the result of two distinct steps. In the first step agent n, using a set of estimators $\{g_{n,1}, g_{n,2}, \dots\}$, forms his expectation at time t about the behavior of future prices, for instance expected price return $E_n[r_{t+1}] = g_{n,1}(\mathbb{I}_{t-1})$ or variance $V_n[r_{t+1}] = g_{n,2}(\mathbb{I}_{t-1})$. With these expectations, using a choice function h_n, possibly derived from some maximization procedure, he computes the fraction of the wealth invested in the risky asset $x_{t+1,n} = h_n(E_n[r_{t+1}], V_n[r_{t+1}], \dots)$. Investment function f_n defined in Assumption 2.3 is the result of the composition of estimators $\{g_n\}$ and choice function h_n. This interpretation is both intuitive and common in the economic literature, but, even if perfectly compatible with Eq. (8), it is not required by our framework. As discussed in Ref. 6, one could indeed consider as investment function any smooth function which maps the past returns history into the future investment choice.

In the present analysis, however, mainly to make comparisons with previous investigations easier, we will restrict our analysis to a specific set of estimators, analogous to the ones considered in Refs. 2–5.

3. SINGLE AGENT CASE

In this Section we deal with the simplest situation in which a single agent operates in the market. The main reason to perform such analysis rests in its relevance for the multi-agent case. Indeed, in the setting with two or more heterogeneous traders each generic equilibrium requires, as necessary condition for stability, the stability of a suitably defined single agent equilibrium[‡].

Under the assumptions introduced in Section 2 the dynamics of the model is defined by the following system

$$\begin{cases} x_{t+1} = f(r_t, r_{t-1}, \dots) \\ r_{t+1} = \dfrac{x_{t+1} - x_t + e_t\, x_t\, x_{t+1}}{x_t\,(1 - x_{t+1})} \end{cases} \quad . \tag{9}$$

The system in Eq. (9) remains, in general, infinitely dimensional due to the dependence of f on the entire set of realized returns. We overcome this difficulty by assuming that agents use their knowledge of the entire past history of prices only through few statistical estimators. We will exploit the recursiveness inherent to these estimators to reduce Eq. (9) to a low dimensional system.

In the next Section we present the case in which agent investment function is based on a single prediction about future price return. In Sect. 3.2 the analysis is extended to investment functions which also depend on the estimated variance of returns.

3.1. Forecast by extrapolation of past average return

A simple forecasting rule consists in taking the average of past realized returns as predictor for future returns. In a dynamical setting it seems appropriate to model agents who take into consideration the possibility that the "mood" prevailing on the market may change over time, so that more recent values of the price return could contain more information about future prices than the older ones.

This behavior is obtained if the agent's estimation of future price return $y_t = E[r_t|\mathbb{I}_{t-1}]$ is obtained as an exponentially weighted moving average (EWMA) estimator

$$y_t = (1-\lambda) \sum_{\tau=0}^{\infty} \lambda^\tau\, r_{t-\tau} \quad , \tag{10}$$

where $\lambda \in [0, 1)$ is a sort of "memory" parameter, describing the weights attached to the past realized returns. The weights are declining geometrically in the past, so that the last available observation r_t has the highest weight. The value of λ determines how the relative weights are distributed among recent and older observations. In particular, the value $\lambda = 0$ corresponds to the case of *naïve* forecast, when agent uses the last realized return as predictor for the next period return.

[‡]A second possible application of the single agent analysis is to provide a succinct description of the aggregate properties of a system with many relatively homogeneous agents, see Ref. 7 for a discussion.

The EWMA estimator admits the following recursive definition

$$y_t = \lambda y_{t-1} + (1-\lambda) r_t \quad , \tag{11}$$

so that Eq. (9) can be reduced to the following two-dimensional system

$$\begin{cases} x_{t+1} = f(y_t) \\ y_{t+1} = \lambda y_t + (1-\lambda)\,\dfrac{f(y_t) - x_t + e_t\, x_t\, f(y_t)}{x_t\,(1 - f(y_t))} \end{cases} \quad . \tag{12}$$

The investment share is determined through function f on the base of the forecasted return y_t. Function f is defined on the interval of possible returns $[-1, +\infty)$.

We are interested in the analysis of the so-called *deterministic skeleton* of the system in Eq. (12). That is, we substitute the yield by its mean value $\bar{e}$ in order to obtain the deterministic dynamical system which gives, in a sense, the "average" representation of the stochastic dynamics. The following result characterizes the existence and local stability of the fixed points for such deterministic skeleton and also the nature of the possible bifurcations.

THEOREM 3.1. *Let (x^*, y^*) be a fixed point of the deterministic skeleton of system* (12) *with price return r^* and let the **Equilibrium Market Line** $l(r)$ be the following function*

$$l(r) = \frac{r}{\bar{e} + r} \qquad r \in [-1, \infty) \quad . \tag{13}$$

Then:

- *(i) The equilibrium is feasible, i.e. the equilibrium prices are positive, if either $x^* < 1$ or $x^* \geq 1/(1-\bar{e})$.*
- *(ii) Equilibrium price return r^* and equilibrium investment share x^* satisfy*

$$l(r^*) = f(r^*)\ , \quad x^* = f(r^*) \quad . \tag{14}$$

  The equilibrium value of predictor coincides with the equilibrium price return: $y^* = r^*$.
- *(iii) Equilibrium (x^*, y^*) is (locally) asymptotically stable if*

$$\frac{f'(r^*)}{l'(r^*)}\,\frac{1}{r^*} < \frac{1}{1-\lambda}\, , \qquad \frac{f'(r^*)}{l'(r^*)} < 1 \qquad \textit{and} \qquad \frac{f'(r^*)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -\frac{1+\lambda}{1-\lambda} \quad . \tag{15}$$

- *(iv) Equilibrium (x^*, y^*) undertakes* Neimark-Sacker*,* fold *or* flip *bifurcation if the first, the second or the third inequality in* (15) *turns to equality, respectively.*

Proof. See the proof of Theorem 3.2. ☐

The first item states that economically meaningful equilibria are characterized by values of the average investment share inside the intervals $(-\infty, 1)$ or $[1/(1-\bar{e}), +\infty)$.

The second item characterizes equilibria of Eq. (12). First of all notice that the realized price return at equilibrium r^* coincides with the prediction of the EWMA estimator y^*. This is an important consistency result which must hold for any meaningful economic dynamics. Equation (14) provides a simple geometric characterization of all possible equilibria: they can be obtained as the intersections of investment function f with the Equilibria Market Line l. The restriction on the domain of the EML is the counterpart of the restriction on the investment share x^* mentioned in item *(i)* and derived from Eq. (5) at equilibrium. Notice that EML is made of two branches separated by a vertical asymptote[§] in $-\bar{e}$.

[§]The system defined in Eqs. (6) and (7) cannot have an equilibrium return equal to $-\bar{e}$. Indeed, if the price return offsets the positive dividend yield in the equilibrium, each agent would have constant wealth over time. This would lead to constant price, and, therefore, to zero return.

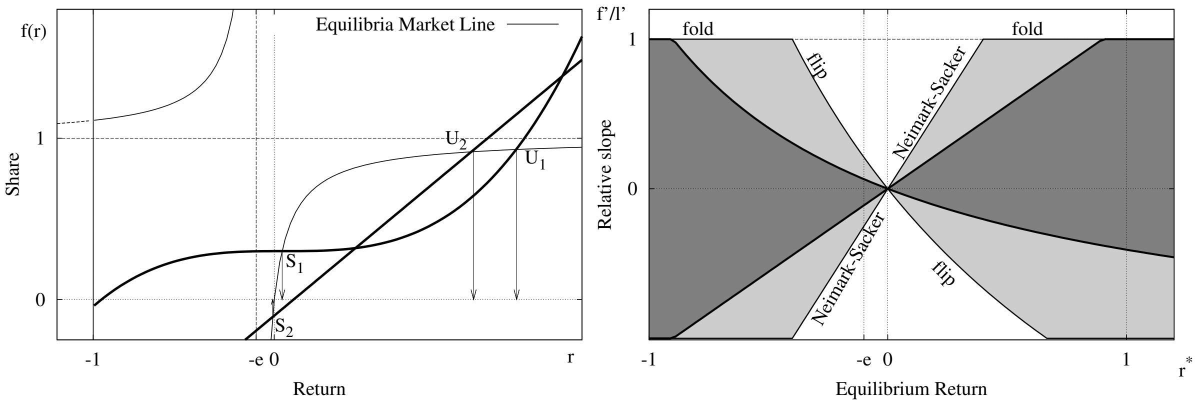

Figure 1. Equilibria and their stability for the system in Eq. (12). Left Panel: the EML and equilibria for two different investment functions. Right Panel: the stability regions in the coordinates r^* and $f'(r^*)/l'(r^*)$ for two different λ's. For small value $\lambda = 0.1$, the fixed point is locally stable in one of two black sets. When λ increases to the value 0.6, the stability area in addition contains two grey areas.

In the left panel of Fig. 1 we show the geometric identification of the equilibria for two different investment functions (thick lines) as intersections with the EML (thin line). The abscissa of any intersection gives the value of return r^* (equal to agent forecast y^*), while the ordinate gives investment share x^*. The nonlinear function in Fig. 1 has two equilibria: S_1 with small positive return and U_1 with high positive return. The linear function has two equilibria too, S_2 and U_2. Even if in S_2 the equilibrium price return is negative, it is greater than $-\bar{e}$, so that the return in terms of unscaled price is still positive. If x^* is positive, the equilibrium return is positive as well, and risky asset price p_t steadily increases. If x^* is negative, the equilibrium price return is negative and the price of the risky asset goes asymptotically to 0. Finally, if $x^* = 0$ the price of the asset is stationary.

The third item of Theorem 3.1 defines the conditions for the asymptotic stability of equilibria. The three relevant parameters are λ, r^* and the relative slope of the investment function w.r.t. the EML $f'(r^*)/l'(r^*)$. The stability regions for two different values of λ in terms of the last two parameters are shown in the right panel of Fig. 1 as differently shaded gray areas.

Notice that if the slope of f at the equilibrium increases, the system tends to lose its stability. In particular, the second inequality in (15) requires the slope of investment function to be smaller than the slope of the EML. Then equilibria U_1 and U_2 in Fig. 1 (left panel) can be immediately recognized as unstable.

Finally, item *(iv)* clarifies what type of bifurcation is encountered when one of the stability conditions in Eq. (15) is violated. The first-order bifurcation types are reported in Fig. 1 (right panel).

3.2. Forecast by extrapolation of past average return and variance

One extension of the previous model consists in considering agents who decide their future investment based not only on the estimated value of future return, but also on the estimated variance. The latter can be thought of as a measure of expected risk.

In order to extend the agent's investment function to consider also some measure of risk, we introduce the exponentially weighted moving average (EWMA) estimator for the variance

$$z_t = (1-\lambda)\sum_{\tau=0}^{\infty}\lambda^\tau\left(r_{t-\tau}-y_t\right)^2 \quad , \tag{16}$$

where y_t is an EWMA of past returns as given in Eq. (10) and $\lambda \in [0,1)$ is the usual "memory" parameter.

Notice that estimators (10) and (16) together admit the following recursive definition

$$\begin{aligned} y_t &= \lambda y_{t-1} + (1-\lambda) r_t \\ z_t &= \lambda z_{t-1} + \lambda\,(1-\lambda)\big(r_t - y_{t-1}\big)^2 \quad , \end{aligned} \tag{17}$$

which allows the reduction of Eq. (9) to a 3-dimensional dynamical system

$$\begin{cases} x_{t+1} = f(y_t, z_t) \\ y_{t+1} = \lambda y_t + (1-\lambda)\,\dfrac{f(y_t,z_t) - x_t + e_t\, x_t\, f(y_t,z_t)}{x_t\,(1 - f(y_t,z_t))} \\ z_{t+1} = \lambda z_t + \lambda(1-\lambda)\left[\dfrac{f(y_t,z_t) - x_t + e_t\, x_t\, f(y_t,z_t)}{x_t\,(1 - f(y_t,z_t))} - y_t\right]^2 \end{cases} . \tag{18}$$

Investment function f depends now on two variables, y_t and z_t, and is generally defined on $[-1,+\infty) \times [0,+\infty)$.

The following Theorem characterizes the fixed points of the deterministic skeleton of system (18) and gives the sufficient conditions for their local stability.

Theorem 3.2. *Let (x^*, y^*, z^*) be a fixed point of the deterministic skeleton of system* (18) *and let r^* be a price return in this point. The feasibility conditions for equilibrium are as provided in item (i) of Theorem 3.1. Moreover, with the same definition for the Equilibria Market Line $l(r)$ one has*

(i) Equilibrium price return r^* and equilibrium invested share x^* satisfy

$$l(r^*) = f(r^*, 0)\,, \qquad x^* = f(r^*, 0)\,. \tag{19}$$

The equilibrium value of the return predictor coincides with the equilibrium price return $y^* = r^*$, while for the predictor of the variance it is $z^* = 0$.

(ii) The fixed point is (locally) asymptotically stable if

$$\frac{f'_y(r^*,0)}{l'(r^*)}\,\frac{1}{r^*} < \frac{1}{1-\lambda}\,, \qquad \frac{f'_y(r^*,0)}{l'(r^*)} < 1 \qquad \text{and} \qquad \frac{f'_y(r^*,0)}{l'(r^*)}\,\frac{2+r^*}{r^*} > -\frac{1+\lambda}{1-\lambda} \quad . \tag{20}$$

(iii) System (18) *undertakes* Neimark-Sacker*,* fold *or* flip *bifurcation, if the first, the second, or the third inequality turns to equality, respectively.*

Proof. See Appendix A.1. ∎

It may come as a surprise, but there are only minor differences between the last statement and Theorem 3.1. In item *(i)* the conditions characterizing the equilibrium investment share and return have slightly changed, because the function f now depends on two variables. However, if one considers the restriction of f on the set $z = 0$, it is clear that the characterization of equilibria as the intersections with the EML given in Section 3.1 is still valid. The consistency result for the equilibrium return estimator $y^* = r^*$ is confirmed and is extended to the variance estimator z^*, whose value at equilibrium becomes zero, as expected for a geometrically increasing price dynamics. The stability conditions given in item *(ii)* are similar to the ones derived in Theorem 3.1. Indeed, the partial derivative of function f is exactly the derivative of its restriction, and therefore inequalities in Eq. (20) are the same as in Eq. (15). As a consequence, both the equilibria and the stability regions look the same as in Fig. 1 and the bifurcation analysis remains unchanged.

In conclusion, even if the introduction of a measure of risk could possibly change the global behavior of the system, the local dynamics in a neighborhood of the fixed points remains essentially the same.

4. HETEROGENEOUS AGENTS

In this Section we consider a market populated by two different agents. As in Section 3.2 the agents' investment decisions will be based on the forecast about future return and its variance obtained through EWMA estimators. The considered agents can differ in two respects: first, they weight past observations with different values of the parameter λ and, second, they translate these different forecasts into their investment decisions using different investment functions f.

The main difference of the heterogeneous setting with respect to the single agent case concerns the role of the wealth dynamics. It is convenient to rewrite the evolution of wealth in Eq. (7) in terms of shares as follows

$$\varphi_{t+1,1} = \varphi_{t,1} \frac{1 + x_{t,1}\,(r_{t+1} + e_t)}{1 + \big(x_{t,1}\,\varphi_{t,1} + x_{t,2}\,(1 - \varphi_{t,1})\big)\,(r_{t+1} + e_t)} \quad , \tag{21}$$

where $\varphi_{t,1} = w_{t,1}/(w_{t,1} + w_{t,2})$ stands for the fraction of total wealth owned by agent 1 at time t.

Each agent n ($n \in \{1, 2\}$) forms his expectations about future price return $y_{t,n}$ and variance $z_{t,n}$ using the EWMA estimators defined in Eqs. (10) and (16) with $\lambda = \lambda_n$ and, following his investment function f_n, computes the share of wealth invested in the risky asset $x_{t+1,n}$ as

$$x_{t+1,n} = f_n(y_{t,n}, z_{t,n}) \quad . \tag{22}$$

Functions f_1 and f_2 are two arbitrary continuous functions defined on $[-1, +\infty) \times [0, +\infty)$.

Using the last two equations and Eq. (17) one gets the following 7-dimensional system of difference equations describing the evolution of the market[¶]:

$$\begin{cases} x_{t+1,1} = f_1(y_1, z_1) \\ x_{t+1,2} = f_2(y_2, z_2) \\ y_{t+1,1} = \lambda_1\, y_1 + (1 - \lambda_1)\, r_{t+1} \\ y_{t+1,2} = \lambda_2\, y_2 + (1 - \lambda_2)\, r_{t+1} \\ z_{t+1,1} = \lambda_1\, z_1 + \lambda_1\,(1 - \lambda_1)\,\big(r_{t+1} - y_1\big)^2 \\ z_{t+1,2} = \lambda_2\, z_2 + \lambda_2\,(1 - \lambda_2)\,\big(r_{t+1} - y_2\big)^2 \\ \varphi_{t+1,1} = \varphi_1 \dfrac{1 + x_1\,(r_{t+1} + e_t)}{1 + (x_1\,\varphi_1 + x_2\,(1 - \varphi_1))\,(r_{t+1} + e_t)} \end{cases} \quad , \tag{23}$$

where from Eq. (6) the price return r_{t+1} is given by

$$r_{t+1} = \frac{\varphi_1\,\big((1 + e_t\,x_1)\,f_1(y_1, z_1) - x_1\big) + (1 - \varphi_1)\,\big((1 + e_t\,x_2)\,f_2(y_2, z_2) - x_2\big)}{\varphi_1\,x_1\,\left(1 - f_1(y_1, z_1)\right) + (1 - \varphi_1)\,x_2\,\left(1 - f_2(y_2, z_2)\right)} \quad . \tag{24}$$

As in the single agent case we analyze the deterministic skeleton of Eq. (23) replacing yield process $\{e_t\}$ by its mean value $\bar{e}$. We will make use of the following

Definition 4.1. *Let $\boldsymbol{x}^*$ stands for a fixed point of the deterministic skeleton of system 23. Agent n is said to "survive" in $\boldsymbol{x}^*$ if his equilibrium wealth share is strictly positive, $\varphi^*_n > 0$, while he is said to "dominate" the economy if he possesses, at equilibrium, the entire wealth $\varphi^*_n = 1$.*

The asymptotic dynamics of the deterministic skeleton of Eq. 23 is provided by the following theorem which can be easily proven with a bit of algebra

Theorem 4.2. *Let $\boldsymbol{x}^* = (x^*_1, x^*_2, y^*_1, y^*_2, z^*_1, z^*_2, \varphi^*_1)$ be a fixed point of the deterministic skeleton of system in Eq. (23), let r^* be the price return in this point and $l(r)$ the Equilibrium Market Line defined above, then it is*

$$r^* = y^*_1 = y^*_2 \qquad \text{and} \qquad z^*_1 = z^*_2 = 0 \quad . \tag{25}$$

and the following two cases are possible

[¶]In order to avoid the overloading of the notation from now on we leave out the indexes t from the variables in the right-hand sides of the equations.

(i) ***Single agent survival.*** *In* $\boldsymbol{x^*}$ *only one agent possesses a wealth share different from zero. Without loss of generality we can assume this agent to be agent* 1*, so that* $\varphi_1^* = 1$*. Equilibrium return* r^* *and equilibrium invested shares* x_1^* *and* x_2^* *satisfy*

$$l(r^*) = f_1(r^*, 0) \ , \quad x_1^* = f_1(r^*, 0) \ , \quad x_2^* = f_2(r^*, 0) \ . \tag{26}$$

(ii) ***Two agents survival.*** *In* $\boldsymbol{x^*}$ *both agents possess positive wealth shares, so that* $\varphi_1^* \in (0, 1)$*. In this case the equilibrium return* r^* *must simultaneously satisfy the following set of equations*

$$l(r^*) = f_1(r^*, 0) = x_1^* = x_2^* = f_2(r^*, 0) \quad . \tag{27}$$

This Theorem distinguishes between two substantially different cases. In the first case, when a single agent survives, Theorem 4.2 defines a precise value for each component of the equilibrium $\boldsymbol{x^*}$, so that a single point is uniquely determined. In the second case, on the contrary, there is a residual degree of freedom in the definition of the equilibrium: the equilibrium wealth shares of agents may be arbitrary, as long as $\varphi_1^* + \varphi_2^* = 1$. Consequently, item *(ii)* does not define a single equilibrium point but an equilibria hyperplane in the parameter space. The particular fixed point eventually chosen by the system will depend on the initial conditions.

The two cases of Theorem 4.2 also differ in their ability to account for "typical" situations. Indeed, while in the first case no requirements are imposed on the behavior of the investment function of the two agents, the two survivors equilibria require that the values of the investment functions of the agents are, at equilibrium, the same. Consequently, the two-survivors equilibria are non-generic.

Notice that both types of equilibria derived in Theorem 4.2 are strictly related to single-agent equilibria. The determination of the equilibrium return level r^* for the two agents case in Eqs. (26) or (27) is identical to the case where the agent, or one of the agents, who would survive in the two-agent equilibrium, is present alone in the market. An useful consequence of this fact is that the geometrical interpretation of market equilibria presented in Section 3.1 can be extended to illustrate how equilibria with two agents are determined.

As an example consider again the left panel in Fig. 1 and suppose that two strategies shown there belong to two agents who are simultaneously operating on the market. According to Theorem 4.2 all possible equilibria can be found as the intersections of one of the strategy with the Equilibria Market Line (c.f. Eqs. (26) and (27)). In this example there are four possible equilibria. In two of them (S_1 and U_1) the first agent, with non-linear strategy, survives so that $\varphi_1^* = 1$, while in the other two equilibria (S_2 and U_2) the second agent survives and $\varphi_1^* = 0$. In each equilibrium, the intersection of the investment function of the surviving agent with the Equilibria Market Line gives both equilibrium return r^* and the equilibrium investment share of the survivor. The equilibrium investment share of the other agent can be found, accordingly to Eq. (26), as the intersection of its own investment function with the vertical line passing through the equilibrium return. Since the two investment functions shown in Fig. 1 do not possess common intersections with the EML, in this case equilibria with two survivors are impossible.

Among the different equilibria which were characterized in Theorem 4.2, which is the one actually chosen by the market? To answer this question, one has, first of all, to perform the stability analysis of the different possible fixed points. The results of this analysis are collected in the following

Theorem 4.3. *Let* $\boldsymbol{x}^* = (x_1^*, x_2^*, r^*, r^*, 0, 0, \varphi_1^*)$ *be a fixed point of the system in Theorem 4.2. Then*

(i) If $\exists n \in \{1, 2\}$ *s.t.* $\varphi_n^* = 1$*, then the fixed point is (locally) asymptotically stable if*

$$x_{n'}^* < x_n^* \quad n' \neq n \qquad \text{and} \qquad \frac{f'_{n,y}}{x_n^*\,(1 - x_n^*)} < \frac{1}{1 - \lambda_n} \ , \frac{r^*\, f'_{n,y}}{x_n^*\,(1 - x_n^*)} < 1 \ , \frac{(2 + r^*)\, f'_{n,y}}{x_n^*\,(1 - x_n^*)} > -\frac{1 + \lambda_n}{1 - \lambda_n} \ , \tag{28}$$

where $f'_{n,y} = df_n(y^*, 0)/dy$*.*

(ii) The fixed point with $\varphi_1^* \in (0, 1)$ *is non hyperbolic.*

Proof. See Appendix A.2. □

According to item *(ii)* the case of two survivors corresponds to non hyperbolic equilibria. Then, their stability would require the study of higher order expansions of the mapping around the fixed point. Due to the non-generic nature of these equilibria, however, this analysis would largely fall outside the intent of the present paper.

The stability conditions for a generic fixed point in item *(i)* contains two requirements. On the one hand, the stable equilibrium should be "self-consistent", i.e. it should remain stable even if the non-surviving agent would be removed from the economy. Indeed the last three inequalities in Eq. (28) coincide with the corresponding conditions in Eq. (20). This is however not enough. A further requirement comes from the first inequality in Eq. (28): the surviving agent must be more aggressive than the other, i.e. he must invest higher wealth share in the risky asset.

Let us revert once again to the EML "plot" to obtain a geometrical illustration of the two-agents stability conditions. Consider the two functions on the left panel of Fig. 1 with their four possible equilibria. First of all, the two agents dynamics cannot be attracted by U_1 or U_2. Since these equilibria were unstable in the respective single-agent cases, they cannot become stable when both agents are present in the market. If one assumes that S_1 is a stable equilibrium for the first agent alone and S_2 for the second, then S_1 is the only stable equilibrium of the system with two agents. The point S_2 cannot be stable since the first agent possesses in this point a more aggressive strategy than the second and, consequently, his investment function is above the investment function of the latter.

In what follows we shortly discuss two relevant implications of the Theorems presented above which have particular strong connections with previous investigations. The first question has to do with the analysis of the effect of the various parameters that characterize the investment choice of agents on the stability of equilibria. The second, instead, provides some hints about the "principle" according to which market tends to select the surviving strategies.

4.1. On the Determinants of Equilibria Stability

The Theorems above provide "generic" relationships among a small set of parameters characterizing the structure of the economy and the choices of agents. These relationship can be straightforwardly applied to the discussion of specific functional forms, like the ones analyzed in Refs. 4. Our discussion suggests, however, some generic features which apply irrespectively of the latter.

The comparison between the two cases considered in Section 3, for instance, reveals that consideration of the risk by the agents does not have any impact on the location of market equilibria and on their local stability.

A second result concerns the role of the "memory" parameter λ in the stability of equilibria. Since the locations of the equilibria do not depend on the values of λ's (c.f. Theorem 4.2), it immediately follows from Eq. (28) that the stability domain (in parameters space) increases with survivor's λ. The scope of this statement is however restricted by two important *caveats*. First, if the additional condition in Eq. (28) is not satisfied and the first agent is not as aggressive as the second, the equilibrium remains unstable irrespectively of the value of λ. Second, even if the first agent does behave aggressively, the possibility to stabilize any equilibrium by means of λ_1 is partly a consequence of the chosen EWMA estimator and does not possesses a general character. Indeed, an analogous property is proven in Ref. 8 for the EWMA estimator inside a CARA framework. On the other hand, inside CRRA framework there may be unstable equilibria even with infinite agents' memory, if past observations have, for example, constant weights in the agents' estimators.

4.2. On the Competition among Strategies

An interesting implication of Theorem 4.3 concerns the question, common in evolutionary literature,[9] of competition among different agents.

Consider a stable two-agent equilibrium $\boldsymbol{x}^*$ with return r^* and in which only the first agent survives. According to Eq. (7) the wealth return of the survivor is equal to $\rho_1^* = x_1^* \, (r^* + \bar{e})$. Since the first agent asymptotically possesses the total wealth of the economy, ρ_1^* will also be the asymptotic growth rate of the total wealth. Then, we can interpret the second requirement of item *(i)* of the Proposition as saying that, in the dynamical competition, those agent survives who allows the economy to grow with the highest possible rate. Indeed, if the second

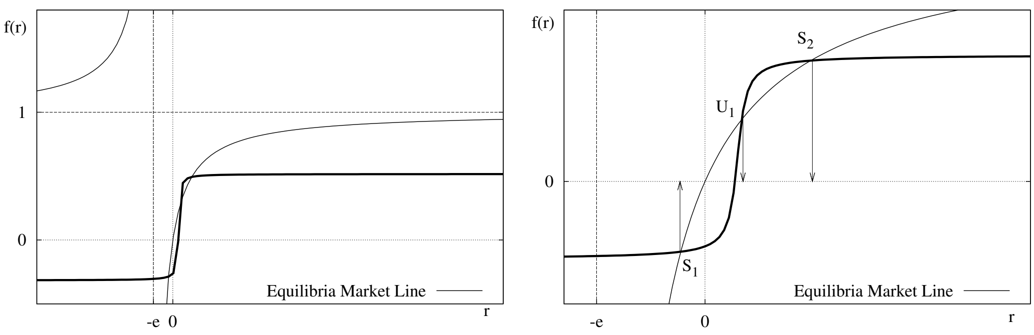

Figure 2. Example of investment function with multiple stable equilibria. Equilibria S_1 and S_2 are stable while U_1 is unstable. The right panel shows a portion of the left panel enlarged.

agent survived, the economy would grow (in this equilibrium) with a rate $x_2^* \left(r^* + \bar{e}\right)$. One could call this result an *optimal selection principle* since it characterizes the market endogenous selection towards the best aggregate outcome.

In a similar framework in Ref. 4 Chiarella and He introduce an analogous principle that has, however, for the particular functional form of their investment functions, much stronger consequences. They found that, when two agents have *linear* strategies, the dynamics endogenously tends to the equilibrium which has relatively higher return (among all "possible" equilibria). Indeed, with some additional care, they add the *quasi* specifier to the word *optimal* to stress that equilibrium is "possible" only if it is stable in the associated single agent case. For instance, in Fig. 1 (left panel) the market select for the better choosing S_1 instead of S_2. However, this choice is only quasi-optimal, since the dynamics does not end up in U_2 where the return is even higher. Even if for the linear strategies considered in Ref. 4 such extended optimal selection principle worked, it will not work in the general setting due to the possibility of *multiple stable equilibria*. One can start, for instance, with an agent whose strategy is not linear and has two stable equilibria like in Fig. 2. Now, any market where such agent meets other agent who always invests less, will have more than one stable equilibrium strategy. Then, the quasi-optimal principle in the sense of Chiarella and He is violated.

The existence of multiple equilibria also leads to another implication: the fact that the dominance of one strategy on another depends on the market initial conditions. Consider the simple case with two agents on the market. Let us suppose that their strategies are such that two stable equilibria exist. In the first equilibrium the first agent dominates, while in the second equilibrium he is dominated by the second agent. Now assume that these two agents enter the market subsequently. It is immediate to see that it is *the order* in which these two agents enter the market that determines the final aggregate outcome.

5. CONCLUSIONS

This paper extends previous contributions and presents novel results concerning the characterization and the stability of equilibria in speculative pure exchange economies with heterogeneous traders.

We considered a simple analytical framework using a minimal number of assumptions (2 assets and Walrasian price formation). We modeled agents as speculative traders and we imposed the constraint that their participation to the trading activity is described by an individual demand functions proportional to their wealth. We found that with prescribed but arbitrary specification about the agents behavior, the feasible dynamics of the economy (i.e. the dynamics for which prices stay always positive) can be described as a multi-dimensional dynamical system. For definiteness and in order to provide stricter characterization of the properties of the model, following a common approach in the agent-based literature, we considered agents who form their individual demands using EWMA predictors of future price returns and variance based on the publicly available past prices history.

Inside this framework, we analyzed the case when a single agent operates in the market, characterizing all possible equilibria and their stability conditions. We also discussed the effect on equilibria stability of the agents "memory", i.e. of the length of the past market history that agents take into consideration to build their predictions about future prices.

Then, we moved to the case of 2 heterogeneous agents and presented the fixed point stability analysis of this system. We showed that the conditions for the existence of fixed points and the conditions for their stability are very similar to the corresponding conditions in the situation with one single agent. Notwithstanding this similarity, however, we found that different scenarios are possible: in stable fixed points, one can have a non-generic case where two traders coexist on the market and a generic case in which one of the agents dominates the other and ultimately captures the entire market. However, it can also be the case that two agents, whose strategies, when present alone on the market, lead to the systems possessing stable fixed points, lead, when present together, to a system that does not possess any stable fixed point. Or, more interesting, they can lead to a system with multiple stable fixed points. This last possibility implies that the final dominance of one strategy on the other depends on the market initial conditions. Ultimately, this is a proof of the impossibility (at least inside our framework) to build any dominance order relation on the space of trading strategies.

The analysis presented in this paper can be extended in many directions. First of all, inside our general framework, numerous specifications of the traders strategies are possible, in addition to the ones considered here. For instance, in Ref. 6 we present a more general analysis where any smooth functions which map the past returns history into the future investment choice are allowed as agents investment functions. But the range of possible extensions encompasses also other interesting cases like some form of evaluation of the "fundamental" value of the asset, maybe obtained from a private source of information, or a strategic behavior that tries to keep into consideration the reaction of other market participants to the revealed individual choices. The same can be said about the dividend yield process, that we assumed randomly and independently drawn from a stationary distribution. Actually, this assumption implies that the investors are not aware whether the price is growing because of some fundamental reasons, or because of a non-fundamental speculation-driven price bubble. It would be interesting to relax our assumptions on the yield process and consider agents who try to distinguish between these two different situations and behave correspondingly.

APPENDIX A. PROOFS OF THEOREMS

This Section present short proofs of the main Theorems of the paper. For more detailed derivations we refer the reader to Ref. 7.

A.1. Proof of Theorem 3.2

Item *(i)* can be easily obtained by the direct substitution of the equilibrium values into Eq. (18). In order to get item *(ii)*, consider the Jacobian matrix in (x^*, y^*, z^*)

$$J = \begin{bmatrix} 0 & f'_y(y^*, z^*) & f'_z(y^*, z^*) \\ -(1-\lambda)/a & \lambda + (1-\lambda)\,(1+y^*)\,f'_y(y^*, z^*)/a & (1-\lambda)\,(1+y^*)\,f'_z(y^*, z^*)/a \\ 0 & 0 & \lambda \end{bmatrix} \quad ,$$

where $a = x^*\,(1-x^*)$ and f'_y and $f'z$ are the partial derivatives, computed in (y^*, z^*), of function f with respect to y and z, respectively. One of the eigenvalues is $\lambda < 1$, while the others are the eigenvalues of the upper-left 2×2 matrix (notice that this is the Jacobian matrix of system (12)). The stability conditions for a two-dimensional system are well-known: in terms of trace Tr and Det of the Jacobian matrix they read $Det < 1$, $Tr < 1 + Det$ and $Tr > -1 - Det$. The types of bifurcation is determined by the inequality which is violated. Applying these conditions to the upper-left matrix we obtains the conditions in Eq. (20) and the statement in item *(iii)*.

A.2. Proof of Theorem 4.3

Let us consider the Jacobian J of system (23). This 7×7 matrix computed in $\boldsymbol{x}^*$ reads

$$\begin{bmatrix} 0 & 0 & f'_{1,y} & 0 & f'_{1,z} & 0 & 0 \\ 0 & 0 & 0 & f'_{2,y} & 0 & f'_{2,z} & 0 \\ (1-\lambda_1)\, r'_{x_1} & (1-\lambda_1)\, r'_{x_2} & \lambda_1 + (1-\lambda_1)\, r'_{y_1} & (1-\lambda_1)\, r'_{y_2} & (1-\lambda_1)\, r'_{z_1} & (1-\lambda_1)\, r'_{z_2} & (1-\lambda_1)\, r'_{\varphi_1} \\ (1-\lambda_2)\, r'_{x_1} & (1-\lambda_2)\, r'_{x_2} & (1-\lambda_2)\, r'_{y_1} & \lambda_2 + (1-\lambda_2)\, r'_{y_2} & (1-\lambda_2)\, r'_{z_1} & (1-\lambda_2)\, r'_{z_2} & (1-\lambda_2)\, r'_{\varphi_1} \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ \varphi'_{x_1} & \varphi'_{x_2} & \varphi'_{y_1} & \varphi'_{y_2} & \varphi'_{z_1} & \varphi'_{z_2} & \varphi'_{\varphi_1} \end{bmatrix}$$

where $f'_{n,y}$ and $f'_{n,z}$ stand for the partial derivatives of the function f_n $(n = 1, 2)$ with respect to y and z and where r'_{x_n} $[r'_{y_n},\, r'_{z_n},\, r'_{\varphi_1}]$ and φ'_{x_n} $[\varphi'_{y_n},\, \varphi'_{z_n},\, \varphi'_{\varphi_1}]$ denote the partial derivatives of r_{t+1} and $\varphi_{t+1,1}$, with respect to $x_{t,n}$ $[y_{t,n},\, z_{t,n}$ and $\varphi_{t,1}]$. The zero entries in the 5-th and 6-th rows are due to the squared term in the expressions of $z_{t+1,1}$ and $z_{t+1,2}$ in Eq. (23). The derivative of this term with respect to any variable will contain the factor $r_{t+1} - y_{t,i}$, which is equal to 0 in the fixed point.

It is obvious that λ_1 and λ_2 are two eigenvalues of the Jacobian, and both are inside the unit circle. The remaining matrix can be further simplified with the computations of the different partial derivatives in the fixed points. We have

$$r'_{\varphi_1} = \frac{\bar{e}\, x_1^*\, x_2^*\, (x_1^* - x_2^*)}{x_2^*\,(1 - x_2^*) + \varphi_1^* \left(x_1^*\,(1 - x_1^*) - x_2^*\,(1 - x_2^*)\right)^2} \quad ,$$

which is equal to 0 in all equilibria with two survivors, because according to item *(ii)* of Theorem 4.2 in this case it is $x_1^* = x_2^*$. Thus, when $\varphi_1^* \in (0, 1)$, the Jacobian has all zeros in the last column, except φ'_{φ_1}.

On the other hand, in the equilibrium with $\varphi_1^* = 1$, it is $\varphi'_{x_1} = \varphi'_{x_2} = 0$. Moreover, with the chain rule one can see easily that $\varphi'_{y_1} = \varphi'_{y_2} = \varphi'_{z_1} = \varphi'_{z_2} = 0$, since $\partial\varphi_{t+1,1}/\partial r_{t+1} = 0$. Therefore, when $\varphi_1^* = 1$, all the elements, except φ'_{φ_1}, are equal to 0 in the last row of the Jacobian.

Let us compute φ'_{φ_1}, which has been proven to be an eigenvalue in all equilibria

$$\varphi'_{\varphi_1} = \frac{1 + x_2^*\,(r^* + \bar{e})}{1 + x_1^*\,(r^* + \bar{e})} \quad . \tag{29}$$

In all equilibria with two survivors, this expression is equal to 1, so that item *(ii)* is proven.

If, instead, $\varphi_1^* = 1$, then the eigenvalue in Eq. (29) is inside the unit circle when $x_1^* > x_2^*$ which proves the first inequality in (28). Now, in this fixed point it is $r'_{x_2} = r'_{y_2} = 0$, so that the Jacobian has eigenvalues λ_2 and 0, which are both inside the unit circle. The last two eigenvalues are the eigenvalues of the matrix

$$\begin{bmatrix} 0 & f'_{1\,y} \\ (1-\lambda_1)\, r'_{x_1} & \lambda_1 + (1-\lambda_1)\, r'_{y_1} \end{bmatrix} \quad .$$

Computing the remaining partial derivatives in the fixed point

$$r'_{x_1} = -\frac{1}{x_1^*\,(1 - x_1^*)} \quad , \qquad \text{and} \quad r'_{y_1} = \frac{f'_{1\,y}\,(1 + y_1^*)}{x_1^*\,(1 - x_1^*)} \quad ,$$

and using the standard conditions for the stability of a two-dimensional systems (which we mentioned in Appendix A.1) one obtains the last three inequalities in Eq. (28).

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