Heterogeneous Beliefs in an Asset Pricing Model with Endogenous Fundamentals

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Introduction

Heterogeneous Agent Models of financial markets

- co-existence of different trading strategies (fundamentalists and chartists, trend-followers) in financial markets
- reproduce several stylized facts of financial markets
- generate endogenous booms and busts, i.e., persistent deviations of price from the fundamental price
- feedback from the relative success of trend-following strategy to price dynamics

Idea of this paper:

- introduce an additional feedback: from past market volatility to the fundamental price
- market crushes can be very severe, they can be followed by long undervaluation of the assets

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CARA framework with heterogeneous expectations

1. two assets
   - riskless: risk-free interest rate $r + 1 = R$
   - risky: price p_t and dividend y_t
     supply per investor $S \geq 0$
2. mean-variance demand

$$A_{h,t}(p) = \left(\mathrm{E}_{h,t}[p_{t+1} + y_{t+1}] - Rp\right) \Big/ \left(a\ \mathrm{V}_{h,t}[p_{t+1} + y_{t+1} - Rp]\right)$$

3. temporary equilibrium

$$S = \sum_h n_{h,t} A_{h,t}(p) \quad \rightsquigarrow \quad p_t \quad \rightsquigarrow \quad R_t = p_t + y_t - Rp_{t-1}$$

4. evolution of fractions

$$n_{h,t+1} \propto \exp(\beta \pi_{h,t}), \quad \text{where profit of type } h \text{ is} \quad \pi_{h,t} = R_t A_{h,t-1} - C_h$$

$\beta \in [0, \infty)$ is the intensity of choice

Fundamental Price

- The fundamental price is the solution under rational (homogeneous) expectations

$$\mathrm{E}_t[p_{t+1} + y_{t+1}] - Rp_t = a\,S\,\mathrm{V}_t[p_{t+1} + y_{t+1} - Rp_t]$$

- For IID dividends, $y_t \sim N(y, \sigma_y^2)$

$$p_{t+1} + y - Rp_t = a\,S\,\mathrm{V}[p_{t+1} + y_{t+1} - Rp_t]$$

and the unique non-bubble solution is

$$p^f = \frac{y}{r} - \frac{aS}{r}\sigma_y^2$$

- present value of the expected dividends adjusted for risk

Conditional Fundamental Price

Conditional fundamental price is the price which fundamentalists expect to be at the "rational" market given past price history

- adjust price given the forecast for the return variance $\hat{\sigma}^2_{R_t|I_{t-1}}$
- for IID dividends, $y_t \sim N(y, \sigma_y^2)$

$$\mathrm{E}^f_{t-1}[p_t] = \frac{y}{r} - \frac{a\,S}{r}\hat{\sigma}^2_{R_t|I_{t-1}}$$

- present value of the expected dividends adjusted for perceived risk
- forward solution of the equilibrium equation under assumption that the demand of all traders is given by

$$A_t(p_t) = \frac{\mathrm{E}[p_{t+1} + y_{t+1}] - Rp_t}{a\hat{\sigma}^2_{R_t|I_{t-1}}}$$

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$$A_t(p_t) = \frac{\mathrm{E}[p_{t+1} + y_{t+1}] - Rp_t}{a\hat{\sigma}^2_{R_t|I_{t-1}}}$$

Dynamical System

- Temporary equilibrium:

$$p_t = \frac{1}{R}\left(\sum_{h=1}^{H} n_{h,t}\, \mathrm{E}_{h,t}[p_{t+1} + y_{t+1}] - S\, a\, \hat{\sigma}^2_{R_t|I_{t-1}}\right)$$

- Evolution of fractions:

$$n_{h,t+1} = \exp(\beta \pi_{h,t}) \Big/ \sum_{h'=1}^{H} \exp(\beta \pi_{h',t})$$

$$\pi_{h,t} = (p_t + y_t - R p_{t-1}) \frac{\mathrm{E}_{h,t-1}[p_t + y_t] - R p_{t-1}}{a\, \hat{\sigma}^2{}_{R_{t-1}|I_{t-2}}} - C_h$$

- EWMA estimation of variance of return

$$\mu_t = (1 - w_\mu) R_t + w_\mu \mu_{t-1}$$

$$\hat{\sigma}^2_{R_{t+1}|I_t} = (1 - w_\sigma)(R_t - \mu_t)^2 + w_\sigma \hat{\sigma}^2_{R_t|I_{t-1}}$$

where $R_t = p_t + y_t - R p_{t-1}$ is the realised excess return

Model of Brock and Hommes, JEDC 1998

Assumptions:

- zero outside supply: $S = 0$ $\rightsquigarrow$ constant fundamental price
- constant expectations of variance: $\mathrm{V}_{h,t} = \sigma^2$

Dynamics:

- Price: $p_t = \frac{1}{R} \sum_{h=1}^{H} n_{h,t} \, \mathrm{E}_{h,t}[p_{t+1} + y_{t+1}] - S \, a \, \hat{\sigma}^2_{R_t | I_{t-1}}$
- Fractions: $n_{h,t+1} = \exp(\beta \pi_{h,t}) \big/ \sum_{h'=1}^{H} \exp(\beta \pi_{h',t})$
- Profit: $\pi_{h,t} = (p_t + y_t - R p_{t-1}) \frac{\mathrm{E}_{h,t-1}[p_t + y_t] - R p_{t-1}}{a \hat{\sigma}^2_{R_{t-1} | I_{t-2}}} - C_h$

Feedbacks:

- past prices affect expectations of chartists
- last period price affects profits and fractions of different types
- expectations and fractions affect current price

Model of Brock and Hommes, JEDC 1998

Assumptions:

- zero outside supply: $S = 0$ $\rightsquigarrow$ constant fundamental price
- constant expectations of variance: $V_{h,t} = \sigma^2$

Dynamics:

- Price: $p_t = \frac{1}{R} \sum_{h=1}^{H} n_{h,t} \, E_{h,t}[p_{t+1} + y_{t+1}] - S \, a \, \hat{\sigma}^2_{R_t | I_{t-1}}$
- Fractions: $n_{h,t+1} = \exp(\beta \pi_{h,t}) \Big/ \sum_{h'=1}^{H} \exp(\beta \pi_{h',t})$
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Feedbacks:

- past prices affect expectations of chartists
- last period price affects profits and fractions of different types
- expectations and fractions affect current price

Model of Gaunersdorfer, JEDC 2000

Assumptions:

- zero outside supply: $S = 0$ $\rightsquigarrow$ constant fundamental price
- time-varying expectations of variance:

$$\mu_t = (1 - w_\mu) R_t + w_\mu \mu_{t-1}$$
$$\hat{\sigma}^2_{R_{t+1}|I_t} = (1 - w_\sigma)(R_t - \mu_t)^2 + w_\sigma \hat{\sigma}^2_{R_t|I_{t-1}}$$

Dynamics:

- Price: $p_t = \frac{1}{R} \sum_{h=1}^{H} n_{h,t} \, \mathrm{E}_{h,t}[p_{t+1} + y_{t+1}] - S \, a \, \hat{\sigma}^2_{R_t|I_{t-1}}$
- Fractions: $n_{h,t+1} = \exp(\beta \pi_{h,t}) \big/ \sum_{h'=1}^{H} \exp(\beta \pi_{h',t})$
- Profit: $\pi_{h,t} = (p_t + y_t - R p_{t-1}) \frac{\mathrm{E}_{h,t-1}[p_t + y_t] - R p_{t-1}}{a \hat{\sigma}^2_{R_{t-1}|I_{t-2}}} - C_h$

Feedbacks:

- the same as in Brock-Hommes

This Model

Assumptions:

- positive outside supply: $S > 0$
- time-varying expectations of variance:
- together they imply time-varying fundamental price

Dynamics:

- Price: $p_t = \frac{1}{R} \sum_{h=1}^{H} n_{h,t} \, \mathrm{E}_{h,t}[p_{t+1} + y_{t+1}] - S\, a\, \hat{\sigma}^2_{R_t|I_{t-1}}$
- Fractions: $n_{h,t+1} = \exp(\beta \pi_{h,t}) \big/ \sum_{h'=1}^{H} \exp(\beta \pi_{h',t})$
- Profit: $\pi_{h,t} = (p_t + y_t - R p_{t-1}) \frac{\mathrm{E}_{h,t-1}[p_t + y_t] - R p_{t-1}}{a \hat{\sigma}^2_{R_{t-1}|I_{t-2}}} - C_h$

Feedbacks:

- the same as in Brock and Hommes
- high past volatility decreases current price
- high past volatility decreases fundamental price, i.e., expectations of fundamentalists

Two-types model: Fundamentalists vs. Chartists

Variance estimate $\sigma^2_{t-1} := \hat{\sigma}^2_{R_t|I_{t-1}}$ using EWMA:

$$\sigma^2_t = (1-w_\sigma)(R_t-\mu_t)^2 + w_\sigma \sigma^2_{t-1}\,, \quad \mu_t = (1-w_\mu)R_t + w_\mu \mu_{t-1}$$

Perceived fundamental price: $p^f_t := \left(y - a\,S\,\sigma^2_{t-1}\right)/r$

Expectations:

- fundamentalists: $\mathrm{E}^f_t[p_{t+1}] = p^f_t$
- chartists: $\mathrm{E}^c_t[p_{t+1}] = p^f_t + g(p_{t-1} - p^f_{t-1}), \qquad g \geq 1$

Model in deviations $x_t := p_t - p^f_t$

- price: $Rx_t = n^f_t\,\mathrm{E}^f_t[x_{t+1}] + n^c_t\,\mathrm{E}^c_t[x_{t+1}] = g n^c_t x_{t-1}$
- fractions: $n^c_{t+1} = 1/(1+\exp(\beta(\pi^f_t - \pi^c_t)))$
- profit differential: $\pi^f_t - \pi^c_t = R_t \frac{-g x_{t-2}}{a\sigma^2_{t-2}} - C$
- excess return: $R_t = x_t - Rx_{t-1} + (y_t - y) + \frac{aS}{r}(R\sigma^2_{t-2} - \sigma^2_{t-1})$

Two-types model: Fundamentalists vs. Chartists

Variance estimate $\sigma_{t-1}^2 := \hat{\sigma}_{R_t|I_{t-1}}^2$ using EWMA:

$$\sigma_t^2 = (1 - w_\sigma)(R_t - \mu_t)^2 + w_\sigma \sigma_{t-1}^2 \,, \quad \mu_t = (1 - w_\mu)R_t + w_\mu \mu_{t-1}$$

Perceived fundamental price: $p_t^f := \left(y - a\,S\,\sigma_{t-1}^2\right)/r$

Expectations:

- fundamentalists: $\mathrm{E}_t^f[p_{t+1}] = p_t^f$
- chartists: $\mathrm{E}_t^c[p_{t+1}] = p_t^f + g(p_{t-1} - p_{t-1}^f), \qquad g \geq 1$

Model in deviations $x_t := p_t - p_t^f$

- price: $Rx_t = n_t^f\,\mathrm{E}_t^f[x_{t+1}] + n_t^c\,\mathrm{E}_t^c[x_{t+1}] = g n_t^c x_{t-1}$
- fractions: $n_{t+1}^c = 1 \big/ \big(1 + \exp\big(\beta(\pi_t^f - \pi_t^c)\big)\big)$
- profit differential: $\pi_t^f - \pi_t^c = R_t \frac{-g x_{t-2}}{a\sigma_{t-2}^2} - C$
- excess return: $R_t = x_t - Rx_{t-1} + (y_t - y) + \frac{aS}{r}(R\sigma_{t-2}^2 - \sigma_{t-1}^2)$

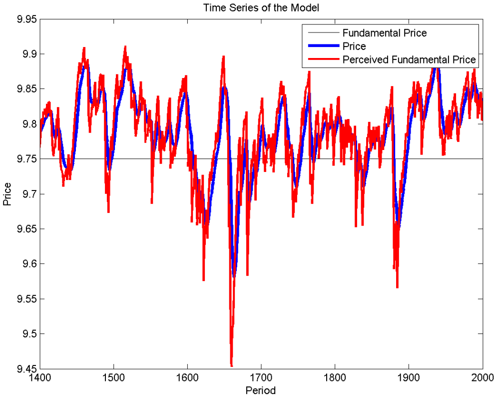
Two-types model: Fundamentalists vs. Chartists

$$
\begin{aligned}
Rx_t &= gn_t^c x_{t-1} \\
n_{t+1}^c &= \left(1 + \exp\left(\beta\left(R_t \tfrac{-gx_{t-2}}{a\sigma_{t-2}^2} - C\right)\right)\right)^{-1} \\
R_t &= x_t - Rx_{t-1} + (y_t - y) + \tfrac{aS}{r}(R\sigma_{t-2}^2 - \sigma_{t-1}^2) \\
\sigma_t^2 &= (1 - w_\sigma)(R_t - \mu_t)^2 + w_\sigma \sigma_{t-1}^2 \\
\mu_t &= (1 - w_\mu)R_t + w_\mu \mu_{t-1}
\end{aligned}
$$

- Deterministic skeleton, $y_t \equiv y$
- Fundamental steady state: $x^* = 0, R^* = 0, \mu^* = 0, \sigma^{2^*} = 0$
- Locally stable steady state for all values of β
- No non-fundamental steady states

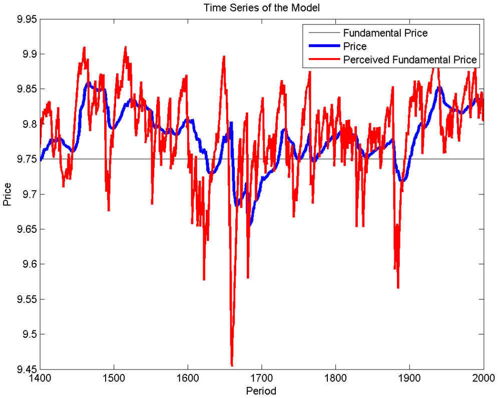
Simulations with stochastic dividend, Low Beta: $\beta = 1$

Parameters: $r = 0.1$, $S = 0.1$, $a = 1$, $g = 1.2$, $w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1$, $\sigma_y = 0.5$



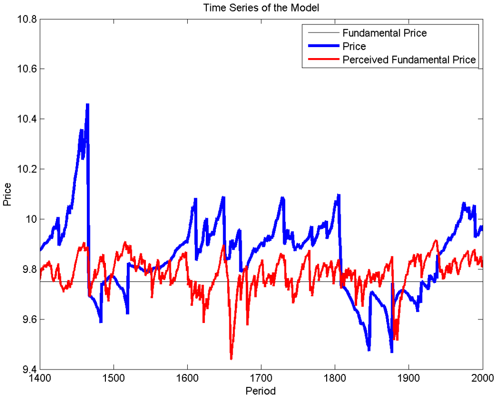
Simulations with stochastic dividend, Low Beta: $\beta = 2$

Parameters: $r = 0.1$, $S = 0.1$, $a = 1$, $g = 1.2$, $w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1$, $\sigma_y = 0.5$



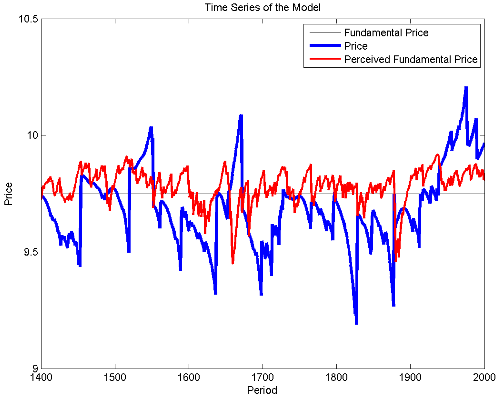
Simulations with stochastic dividend, High Beta: $\beta = 3.5$

Parameters: $r = 0.1$, $S = 0.1$, $a = 1$, $g = 1.2$, $w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1$, $\sigma_y = 0.5$



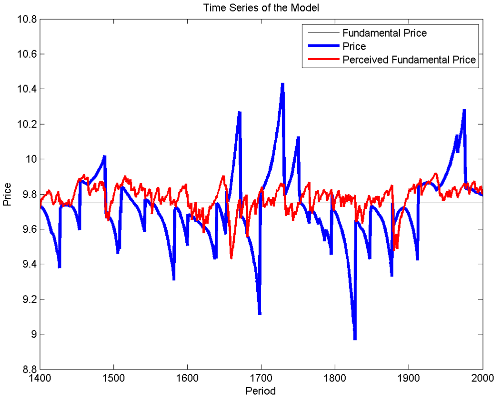
Simulations with stochastic dividend, High Beta: $\beta = 5$

Parameters: $r = 0.1$, $S = 0.1$, $a = 1$, $g = 1.2$, $w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1$, $\sigma_y = 0.5$



Simulations with stochastic dividend, High Beta: $\beta = 10$

Parameters: $r = 0.1, S = 0.1, a = 1, g = 1.2, w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1, \sigma_y = 0.5$



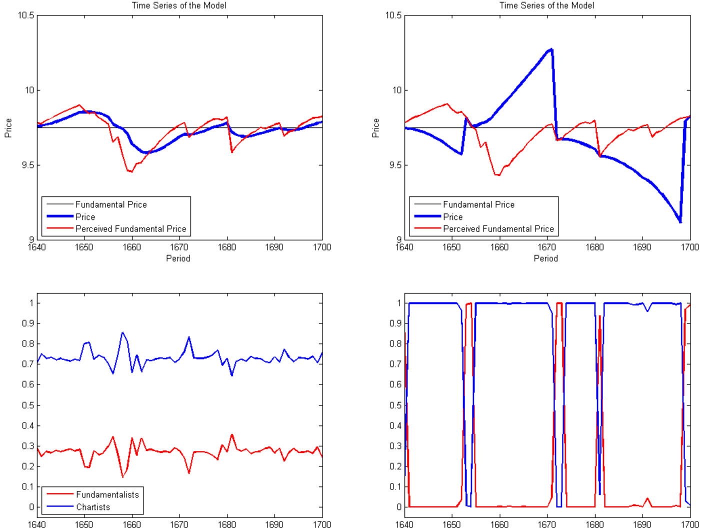
Variance of price vs. variance of fundamental price

Parameters: $r = 0.1, S = 0.1, a = 1, g = 1.2, w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1, \sigma_y = 0.5$

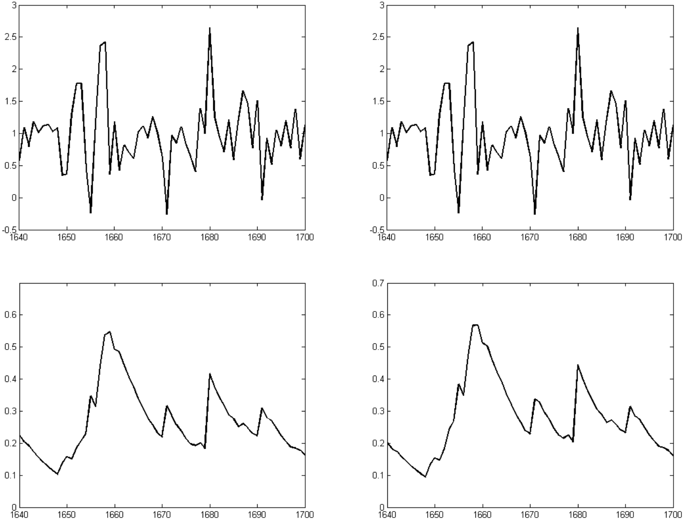
Variance over 1000 periods after 1000 transient periods (one simulation)

Beta	**Dividend**	**Fundamental Price**	**Price**
$\beta = 1$	0.2608	0.0051	0.0033
$\beta = 2$	0.2608	0.0052	0.0016
$\beta = 3.5$	0.2608	0.0053	0.0202
$\beta = 5$	0.2608	0.0058	0.0320
$\beta = 10$	0.2608	0.0055	0.0407

Simulations with stochastic dividend



Simulations with stochastic dividend: Variance



Summary of a simple model

- dynamics is consistent with EMH for low β
  - demand for the risky asset falls as price exhibit high fluctuations
- dynamics exhibit excess volatility for high β
  - price deviations
- endogenous switches between positive and negative attractors (without external noise)
- dynamics of fundamental price is driven by the dividends and slightly amplified by the price
- large price change (in any direction) leads to a drop of fundamental price

Alternative form of the same model

Variance of price estimate $\sigma^2_{t-1} := \hat{\sigma}^2_{p_t|I_{t-1}}$ using EWMA:

$$\sigma^2_t = (1 - w_\sigma)(p_t - \mu_t)^2 + w_\sigma \sigma^2_{t-1}\,, \quad \mu_t = (1 - w_\mu)p_t + w_\mu \mu_{t-1}$$

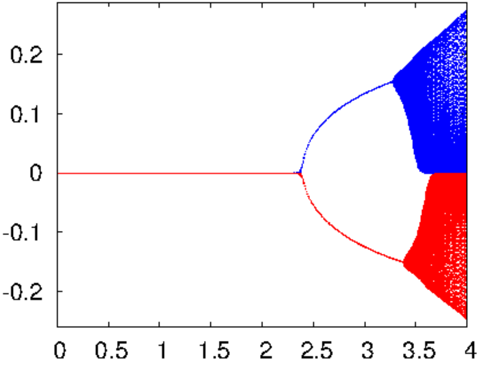
Perceived fundamental price: $p^f_t := \left(y - a\,S\,(\sigma^2_y + \sigma^2_{t-1})\right)/r$

Expectations:

- fundamentalists: $\mathrm{E}^f_t[p_{t+1}] = p^f_t$
- chartists: $\mathrm{E}^c_t[p_{t+1}] = p^f_t + g(p_{t-1} - p^f_{t-1}), \qquad g \geq 1$

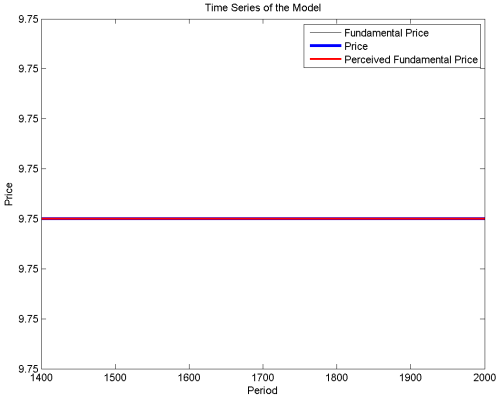
Bifurcation Diagram

Parameters: $r = 0.1, S = 0.1, a = 1, w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1, \sigma_y = 0$



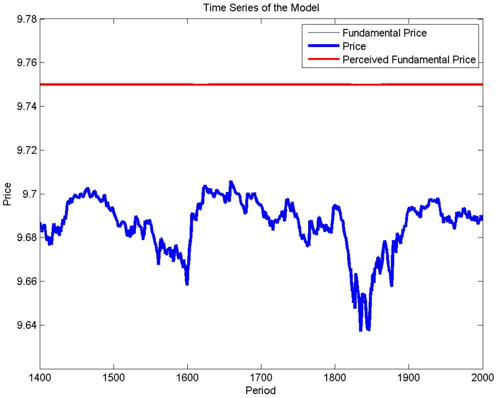
Simulations with stochastic dividend, Low Beta: $\beta = 2$

Parameters: $r = 0.1, S = 0.1, a = 1, w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1, \sigma_y = 0.5$



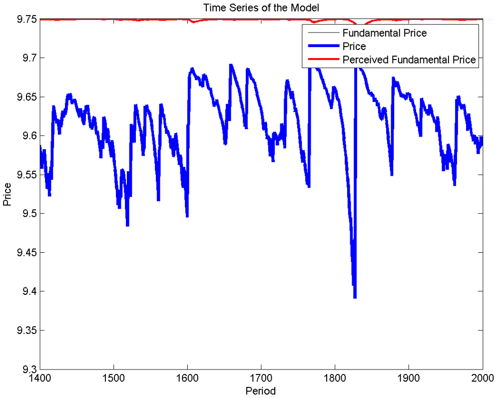
Simulations with stochastic dividend, Middle Beta: $\beta = 2.5$

Parameters: $r = 0.1$, $S = 0.1$, $a = 1$, $w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1$, $\sigma_y = 0.5$



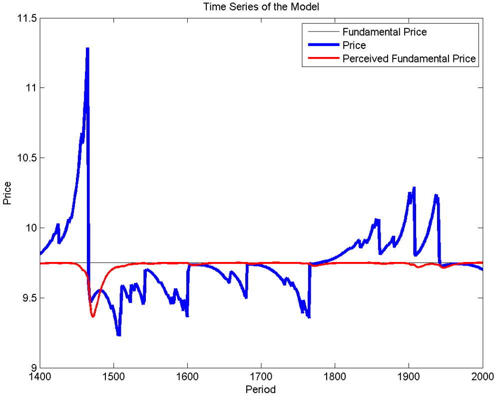
Simulations with stochastic dividend, Higher Beta: $\beta = 3$

Parameters: $r = 0.1$, $S = 0.1$, $a = 1$, $w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1$, $\sigma_y = 0.5$



Simulations with stochastic dividend, High Beta: $\beta = 5$

Parameters: $r = 0.1$, $S = 0.1$, $a = 1$, $w_\mu = w_\sigma = 0.9$
i.i.d. dividends $y = 1$, $\sigma_y = 0.5$



Alternative Dividend Process

AR(1) dividend

$$y_t = y + \rho(y_{t-1} - y) + \varepsilon_t, \quad \text{where} \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

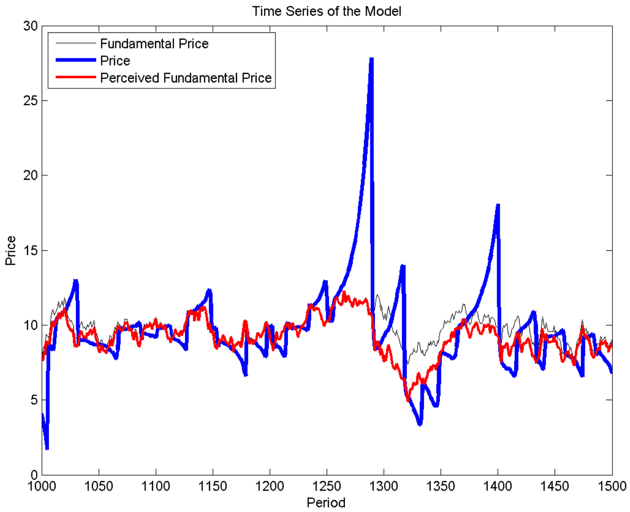
- Fundamental Price

$$p_t^f = \frac{\rho}{R-\rho} y_t + \frac{R}{(R-\rho)r}\left((1-\rho)y - a\,S\frac{R}{R-\rho}\sigma_\varepsilon^2\right)$$

- Perceived Fundamental Price

$$\mathrm{E}_{t-1}^f[p_t] = \frac{\rho}{R-\rho} y_t + \frac{1+r}{(R-\rho)r}\left((1-\rho)y - a\sigma_e^2\right)$$

Consequences of the bubble's burst



Summary

- simple model with fundamentalists and chartists
- agents estimate variance on the past data and update the perceived fundamental price
- creates additional feedback: from past price through volatility to the fundamental price
- after bubble busts, perceived fundamental price may stay lower than the actual fundamental price for a long period of time