Heterogeneous Agent Models
Lecture 1

Introduction
Rational vs. Agent Based Modelling
Heterogeneous Agent Modelling

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Outline

1. Overview
2. Financial Market Model
3. Heterogeneous Agent Models
Economics as Expectation Feedback System

- Expectations play utmost role in any human activity
  - where and when to go to a vacation
  - choice of university degree and specific courses
  - when to buy a car, house, etc.
  - investment choice

- Economics is an expectation feedback system
  - expectations affect people’s decisions
  - individual decisions are aggregated
  - aggregate variables affect expectations
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Cobweb ("hog cycle") Model

- market for a **non-storable** consumption good
- **production lag**: producers form price expectations one period ahead
- **temporary equilibrium**: market clearing prices at each time step

What will be the price at this market?

- **dynamics**: prices evolve over time forming a **trajectory**
- how does this trajectory look like? Is it "simple" or not?
- dynamics depend on functional form of demand and supply and on the way **how expectations are formed**
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Naive Expectations in Cobweb Model

Naive expectations

$$p_t^e = p_{t-1}$$
Price Dynamics

- $p_t^e$: producers' price expectation for period $t$
- $p_t$: realized market equilibrium price
- temporary equilibrium

\[
q_t^d = D(p_t), \quad \text{demand}
\]

\[
q_t^s = S(p_t^e), \quad \text{supply}
\]

\[
q_t^d = q_t^s, \quad \text{market clearing}
\]

\[
p_t^e = H(p_{t-1}, p_{t-2}, \ldots) \quad \text{expectations}
\]

- Price dynamics

\[
p_t = D^{-1}(S(p_t^e)) = D^{-1}(S(H(p_{t-1}, p_{t-2}, \ldots)))
\]
Rational View

- representative agent, who is perfectly rational
- **Rational Expectations**: expectations are model consistent
- **Friedman argument**: “irrational agents will lose money and will be driven out the market by rational agents”
- prices reflect economic fundamentals (**market efficiency**)
Heterogeneous, Interacting Agents Approach

- **heterogeneous** agents, heterogeneous beliefs
- **bounded rationality** (Simon, 1957)
  - market psychology (Keynes, 1936)
  - herding behavior
  - inductive reasoning
  - (imperfect) learning from mistakes
- economy as **complex adaptive, nonlinear, evolutionary** systems
- aggregation of interacting agents leads to non-trivial macro-economic phenomena (**agent-based modelling**)
Some Problems of Interacting Agents Approach

- ‘wilderness’ of bounded rationality
  - many degrees of freedom for heterogeneity
- would not irrational decision makers be driven out?
- non-transparency: what exactly causes the outcome in a (large) computational ABM?

How to Discipline Bounded Rationality?

- stylized agent-based models (Heterogeneous Agent Models)
- behavioral consistency: simple heuristics that work reasonably well
- evolutionary selection (‘survival of the fittest’) and reinforcement learning
- laboratory experiments to test individual decision rules and aggregate macro behavior
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Overview

- **Lecture 1**: Rational vs. agent-based vs. HAM approaches
  - financial market model
- **Lecture 2**: Bifurcation theory and Chaos theory
  - chaos in economics, lessons of nonlinearity for economics
- **Lecture 3**: Learning to Forecast Experiments
  - role of market feedback
- **Lecture 4**: Heuristic Switching Model
  - behavioral model based on evolutionary switching between forecasting heuristics, which explain data very well
Market

Two alternatives for investors, two assets:

Risk-free asset  gross riskless return on the asset \( R = 1 + r > 1 \)

Risky asset  dividend process \( y_t \)
endogenous price \( p_t \) per share

Demand  \( z_t \) for the risky asset is derived from myopic
mean-variance utility maximization

Supply  \( z_s \) per investor is fixed (and assumed to be 0)
Mean-Variaance Demand

- Wealth evolution:

\[ W_{t+1} = R(W_t - p_t z_t) + (p_{t+1} + y_{t+1}) z_t \]

\[ \Downarrow \]

\[ W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t) z_t \]

- Agent solves

\[ \max_{z_t} \left[ E_t W_{t+1} - \frac{a}{2} V_t W_{t+1} \right], \]

where \( a > 0 \) is risk aversion.

- Demand is

\[ z_t = \frac{E_t[p_{t+1} + y_{t+1} - Rp_t]}{a V_t[p_{t+1} + y_{t+1} - Rp_t]} \]
Efficient Market Hypothesis

Eugene Fama (1965), Paul Samuelson (1965)

Market is defined as informationally efficient if

- price $p_t$ of an asset reflects all available (relevant) information
- ...that is price is an unbiased estimation of the aggregate beliefs about the future perspectives (fundamental value $p_t^*$)

Fundamental price is

$$p_t^* = \sum_{k=1}^{\infty} \frac{y_{t+k}}{(1 + r)^k}$$

where $y_t$ is the dividend at time $t$

Formally: $p_t = E_t[p_t^*]$
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Efficient Market Hypothesis

Eugene Fama (1965), Paul Samuelson (1965)

- $p_t = E_t[p_t^*] = p_t^* + \varepsilon_t$
- errors $\varepsilon_t$ are unpredictable on the basis of information available at time $t$
- the returns are unpredictable:
  \[ E_t[p_{t+1} + y_{t+1} - (1 + r)p_t] = 0 \]
- **Question**: Why are the markets efficient?
Theoretical Underpinnings of Efficiency

1st argument: Milton Friedman (1953)

- arbitrage:
  people who argue that speculation is generally destabilizing seldom realize that this is largely equivalent to saying that speculators lose money, since speculation can be destabilizing in general only if speculators on average sell when currency is low in price and buy when it is high

\[
E_t[p_{t+1} + y_{t+1} - (1 + r)p_t] = 0
\]

\[\Downarrow\]

\[1 + r = \frac{1}{p_t} E_t[p_{t+1} + y_{t+1}] \iff p_t = \frac{1}{1 + r} E_t[p_{t+1} + y_{t+1}]\]
Theoretical Underpinnings of Efficiency

2nd argument: Robert Lucas (1978)

- fully rational behavior
  - optimizing behavior of representative investor
  - absence of systematic mistakes about price distribution

- assume demand functions:

\[
 z_{t,n} = \frac{E_{t,n}[p_{t+1} + y_{t+1}] - p_t(1 + r)}{a_n V_{t,n}[p_{t+1} + y_{t+1}]}
\]

- then

\[
 p_t = \frac{1}{1 + r} \sum_n w_{t,n} E_{t,n}[p_{t+1} + y_{t+1}]
\]

where \( w_{t,n} \) is the relative weight on the group “n” of agents.
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Theoretical Underpinnings of Efficiency

2nd argument: Robert Lucas (1978)

- Under Rational Expectations

\[
p_t = \frac{1}{1 + r} \sum_{n} w_{t,n} \mathbb{E}_{t,n}[p_{t+1} + y_{t+1}] \]

\[
\uparrow
\]

\[
p_t = \frac{1}{1 + r} \sum_{n} w_{t,n} \mathbb{E}_t[p_{t+1} + y_{t+1}] \]

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Empirical Consequences

“No-trade theorem”

Instead, we observe persistent trading volume:
Empirical Consequences

Both technical and fundamental analysis are useless (except by luck):

“blindfolded chimpanzee throwing darts at The Wall Street Journal can select a portfolio that performs as well as those managed by the experts”
Empirical Consequences

No-Excess Volatility:

\[ p_t = E_t[p_t^*] = p_t^* + \varepsilon_t \]

therefore

\[ \sigma_{p_t} \leq \sigma_{p_t^*} \]
Empirical Consequences

No-Excess Volatility:

Instead, we observe excess volatility:

Summary of rational view on financial market

- all investors possess rational expectations; they instantaneously discount all available information
- no opportunities for speculative profit
- no place for market psychology or herding
- bubbles and crushes are either non-existent or “rational”
- trading volume is almost zero
- no autocorrelations of returns
Heterogeneity of investors

- assume that traders are not alike

\[ p_t = \frac{1}{1 + r} \sum_n w_{t,n} E_{t,n} [p_{t+1} + y_{t+1}] \]

BUT:

\[ E_{t,n} [p_{t+1}] = \frac{1}{1 + r} E_{t,n} \left[ \sum_m w_{t+1,m} E_{t,m} [p_{t+2} + y_{t+2}] \right] \]

- there is no way to form expectations about others’ expectations about dividends and – even more so – about others’ expectations of price

- consequently, complete knowledge is impossible and agents are boundedly rational
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Heterogeneity of investors

Brian Arthur (1995)

we can think of the economy ultimately as a vast collection of beliefs or hypotheses, constantly being formulated, acted upon, changed and discarded; all interacting and competing and evolving and coevolving; forming an ocean of ever-changing, predictive models-of-the-world

Santa-Fe Artificial Stock Market

Arthur, Holland, LeBaron, Palmer and Tayler (1997)

”Asset Pricing under Endogenous Expectations in an Artificial Stock Market”
Santa-Fe Artificial Stock Market: Setup

- each agent is endowed with multiple forecasting models, $M = 100$
- forecasting model is a predictor
  - predictor checks the market condition and gets “active” if condition is satisfied
    - some of these conditions are “fundamental”
    - some are “technical-trading”
  - predictor also provides two values $a$ and $b$, which imply prediction $E[p_{t+1} + y_{t+1}] = a(p_t + y_t) + b$
  - for each predictor the accuracy (squared forecasting error) is tracked and updated, when the predictor is “active”
- agent uses $H$ most precise “active” predictors to combine them and get a forecast
- on a slower scale (on average every 250 or every 1000 periods) predictors get updated by Genetic Algorithm
Santa-Fe Artificial Stock Market: Results

Two regimes:

- **slow-learning-rate regime**
  - GA is evoked every 1,000 periods in average
  - accuracy of predictors are slowly updated
  - market converges rapidly and then stays in evolutionary stable RE regime, where agents use similar rules, trading volume is low, technical trading does not emerge

- **fast-learning-rate regime**
  - GA is evoked every 250 periods in average
  - accuracy of predictors are updated relatively fast
  - market is often close to the RE solution, but temporary bubbles and crashes emerge systematically, technical trading emerged in the market
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Heterogeneous Agent Models

Brock and Hommes (1998, JEDC) propose a simple model which can reach a similar conclusion.

It also answers the following questions:

- Can heterogeneous agents *destabilize* markets?
- Can “less-rational” traders *survive* against “more-rational”?
- Is the market with heterogeneous agents *efficient*?
- Is it possible to mimic “stylized facts” with a simple (low-dimensional) model?
Model with Heterogeneous Agents

- \( H = 2, 3, \ldots \) types of traders
- **individual demand** is

\[
 z_{h,t} = \frac{E_{h,t}[p_{t+1} + y_{t+1} - Rp_t]}{a V_{h,t}[p_{t+1} + y_{t+1} - Rp_t]}
\]

Assume that agents have...
- heterogeneous expectations about price
- the same risk aversion
- homogeneous expectations of the variance
- complete knowledge of the dividend process (which is IID for simplicity)
Model with Heterogeneous Agents

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Fundamental Solution

Equilibrium pricing equation with homogeneous investors:

\[ Rp_t = E_t[p_{t+1} + y_{t+1}] \]

There exists an unique bounded fundamental solution \( p_t^* \) (discounted sum expected future cash flow):

\[ p_t^* = \frac{E_t[y_{t+1}]}{R} + \frac{E_t[y_{t+2}]}{R^2} + \cdots \]

For a special case of IID dividends, with \( E_t[y_{t+1}] = \bar{y} \):

\[ p_t^* = \frac{\bar{y}}{R - 1} = \frac{\bar{y}}{r} \]

Equilibrium pricing equation with heterogeneous investors:

\[ \sum_{h=1}^{H} n_{h,t} z_{h,t} = 0 \iff p_t = \frac{1}{1 + r} \sum_{h=1}^{H} n_{h,t} E_{h,t}[p_{t+1}] + \frac{\bar{y}}{1 + r} \]
Dynamics in Deviations

- write the equation in the deviations from the rational expectations benchmark \( x_t = p_t - p^* \)

\[
x_t = \frac{1}{1 + r} \sum_{h=1}^{H} n_{h,t} E_{h,t}[x_{t+1}]
\]

- there are two homogeneous, self-fulfilling solutions:
  - if \( E_{h,t}[x_{t+1}] = 0 \) then \( x_t = 0 \) \( \rightsquigarrow \) fundamental solution.
  - if \( E_{h,t}[x_{t+1}] = R^2 x_{t-1} \) then \( x_t = Rx_{t-1} \) \( \rightsquigarrow \) bubble solution.
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Forecasting rules

Assume that belief of type $h$ on future prices has form

$$E_{h,t}[p_{t+1}] = p^* + f_h(x_{t-1}, \ldots, x_{t-L}) \Leftrightarrow E_{h,t}[x_{t+1}] = f_h(x_{t-1}, \ldots, x_{t-L})$$

Important special cases:

- **rational expectations** "$f(x_{t-1}, \ldots, x_{t-L})" = x_{t+1}$
  (assumes perfect foresight on all other belief-fractions $n_{h,t+1}$!)

- **fundamentalists** $f \equiv 0$

- **pure trend chasers** $f(x_{t-1}, \ldots, x_{t-L}) = g x_{t-1}$
  - strong trend chaser: $g > R$
  - contrarian: $g < 0$
  - strong contrarian: $g < -R$

- **pure bias**: $f(x_{t-1}, \ldots, x_{t-L}) = b$. 
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Main Question

Do rational agents and/or fundamentalists drive out trend chasers and biased beliefs?

In BH paper the standard asset pricing model with two, three or four types of agents is investigated

- costly fundamentalists versus cheap trend followers
- fundamentalists versus opposite biases
- fundamentalists versus bias versus trend

Remark: All are described by linear forecast with one lag

\[ f_{h,t} = g_h x_{t-1} + b_h \]
Evolutionary selection of strategies

1. take realized excess return: \( R_t = p_t + y_t - Rp_{t-1} \)

2. recall the demand by type \( h \)

\[
z_{h,t-1} = \frac{E_{h,t-1}[p_t + y_t - Rp_{t-1}]}{a\sigma^2}
\]

3. compute the realized profits in period \( t \) of type \( h \)

\[
\pi_{h,t} = R_t z_{h,t-1} = (p_t + y_t - Rp_{t-1}) \frac{E_{h,t-1}[p_t + y_t - Rp_{t-1}]}{a\sigma^2}
= (x_t - R x_{t-1} + \delta_t) \frac{E_{h,t-1}[x_t - R x_{t-1}]}{a\sigma^2}
\]

with \( y_t = \bar{y} + \delta_t \).
Evolutionary selection of strategies

1. define the **performance measure** as a (weighted sum of) realized profits

   \[ U_{h,t} = \pi_{h,t} + wU_{h,t-1} - C_h \]

   where \( C_h \geq 0 \) are costs for predictor \( h \), and \( w \) is **memory strength**
   - \( w = 1 \): infinite memory; fitness \( \equiv \) accumulated wealth
   - \( w = 0 \): memory is one lag; fitness \( \equiv \) most recently realized net profit

2. update **fractions of belief types** according to the **discrete choice model**

   \[ n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{Z_{t-1}} \]

   where \( Z_{t-1} \) is normalization factor and \( \beta \) is **intensity of choice**.
Asset Pricing Model with Heterogeneous Beliefs

- pricing equation

\[ R_x_t = \sum_{h=1}^{H} n_{h,t} f_h(x_{t-1}, \ldots, x_{t-L}) = \sum_{h=1}^{H} n_{h,t} f_{h,t} \]

- fraction of different investors’ types

\[ n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{k=1}^{H} e^{\beta U_{k,t-1}}} \]

- performance of different types

\[ U_{h,t-1} = (x_{t-1} - R_{x_{t-2}}) \frac{f_{h,t-2} - R_{x_{t-2}}}{a\sigma^2} + C_h \]
Example with linear predictors $f_{h,t} = g_h x_{t-1} + b_h$

- pricing equation
  
  $$x_t = \sum_{h=1}^{H} n_{h,t} (g_h x_{t-1} + b_h)$$

- fraction of different investors’ types
  
  $$n_{h,t} = \frac{e^{\beta U_{h,t-1}}}{\sum_{k=1}^{H} e^{\beta U_{k,t-1}}}$$

- performance of different types
  
  $$U_{h,t-1} = (x_{t-1} - R x_{t-2}) \frac{(g_h x_{t-3} + b_h - R x_{t-2})}{a\sigma^2} - C_h$$
Two-types example: Fundamentalists versus trend-followers

- Two types
  - **fundamentalists**, \( f_{1,t} = 0 \), at cost \( C \)
  - **trend-followers**, \( f_{2,t} = gx_{t-1} \) at cost 0

- Define difference in fractions: \( m_t = n_{1,t} - n_{2,t} \)

- Derive 3-dimensional system

\[
R_{x_t} = n_{2,t}gx_{t-1} \\
mt+1 = \tanh \left( \frac{\beta}{2} \left[ - \frac{gx_{t-2}}{a\sigma^2} (x_{t} - R_{x_{t-1}}) - C \right] \right)
\]
Fundamentalists versus trend-followers

**Theorem** (Existence and stability of fixed points.)

Let $m^{eq} = \tanh(-\beta C/2)$, $m^* = 1 - 2R/g$ and $x^*$ be a positive solution of

$$\tanh\left(\frac{\beta}{2} \left[ \frac{g}{a\sigma^2} (R - 1)(x^*)^2 - C \right] \right) = m^*$$

Then:

- $0 < g < R$ fundamental st-st $E_1 = (0, m^{eq})$ is globally stable
- $g > 2R$ there are three st-st’s: $E_1 = (0, m^{eq})$, $E_2 = (x^*, m^*)$ and $E_3 = (-x^*, m^*)$
- $R < g < 2R$ $E_1 = (0, m^{eq})$ is stable for $\beta < \beta^*$, $E_2 = (x^*, m^*)$ and $E_3 = (-x^*, m^*)$ are stable for $\beta^* < \beta < \beta^{**}$. The system undergoes a pitchfork bifurcation for $\beta = \beta^*$ and Hopf bifurcation for $\beta = \beta^{**}$
Rational Route to Randomness

\[(C = 1, \, g = 1.2, \, r = 0.1, \, \bar{y} = 10)\]

**Corollary:** fundamentalists cannot drive out trend chasers.
Time series immediately after the secondary bifurcation

\[ \beta = 2.81, \ C = 1, \ g = 1.2, \ r = 0.1, \bar{y} = 10 \]
Time series far from the secondary bifurcation

\[ \beta = 4, C = 1, g = 1.2, r = 0.1, \bar{y} = 10 \]
Fundamentalists versus two opposite biased beliefs

- Three types
  - fundamentalists, $f_{1,t} = 0$, at cost 0
  - positive bias, $f_{2,t} = b_2 > 0$ at cost 0
  - negative bias, $f_{3,t} = b_3 < 0$ at cost 0

- Derive 3-dimensional system

$$Rx_t = n_{2,t}b_2 + n_{3,t}b_3$$

$$n_{j,t+1} = \exp\left(\frac{\beta}{a\sigma^2}(b_j - Rx_{t-1})(x_t - Rx_{t-1})\right)/Z_t, \quad j = 1, 2, 3$$
Fundamentalists versus two opposite biased beliefs

Theorem. (Existence and stability steady state.)

- The system has unique fixed point $E$, which equals to the fundamental fixed point when $b_2 = -b_3$.

- $E$ exhibits a Hopf bifurcation for some $\beta = \beta^*$, so that $E$ is stable for $0 < \beta < \beta^*$ and $E$ is unstable for $\beta > \beta^*$. 
Fundamentalists versus two opposite biased beliefs

**Theorem.** (Neoclassical limit, i.e. $\beta = \infty$)
When biased beliefs are exactly opposite, i.e. when $b_2 = -b_3 = b > 0$, then the system has **globally stable 4-cycle**. For all three types, average profit along this 4-cycle is $b^2$.

**Corollary.** Fundamentalists with zero costs and infinite memory can not beat opposite biased beliefs!