

A model of network formation for the overnight interbank market*

Mikhail Anufriev^{a,†} Andrea Deghi^{b,‡} Valentyn Panchenko^{c,§}
Paolo Pin^{d,¶}

^a *Economics Discipline Group, Business School, University of Technology Sydney, Australia*

^b *Department of Political Economy and Statistics, Università di Siena, Italy*

^c *Economics, UNSW Business School, University of New South Wales, Australia*

^d *Department of Decision Sciences and IGIER, Università Bocconi, Italy*

Abstract

We introduce an endogenous network model of the interbank overnight lending market. Banks are motivated to meet the minimum reserve requirements set by the Central Bank, but their reserves are subject to random shocks. To adjust their expected end-of-the-day reserves, banks enter the interbank market, where borrowers decrease their expected cost of borrowing with the Central Bank, and lenders decrease their deposits with the Central bank in attempt to gain a higher interest rate from the interbank market, but face a counter-party default risk. In this setting, we show that a financial network arises endogenously, exhibiting a unique giant component which is at the same time connected but bipartite in lenders and borrowers. The model reproduces features of trading decisions observed empirically in the Italian e-MID market for overnight interbank deposits.

*We benefited from helpful comments by participants of the workshop on Systemic Risk: Mathematical Modelling and Interdisciplinary Approaches in the Isaac Newton Institute for Mathematical Science, Cambridge, UK; of the 20th Annual Workshop on the Economic Science with Heterogeneous Interacting Agents (WEHIA), Sophia Antipolis, France; of the Computing in Economics and Finance, 2015 conference, Taipei, Taiwan, and of the seminars at the University of Queensland, European University at St. Petersburg. This research was supported by the Centre for International Finance and Regulation (project number E233), which is funded by the Commonwealth and NSW Governments and supported by other Consortium members (see www.cifr.edu.au).

[†]*E-mail:* Mikhail.Anufriev@uts.edu.au.

[‡]*E-mail:* andrea.deghi@unisi.it.

[§]*E-mail:* v.panchenko@unsw.edu.au.

[¶]*E-mail:* paolo.pin@unibocconi.it.

1 Introduction

This paper presents an endogenous network formation model of the interbank lending on the overnight market. Theoretical predictions of the model are compared with an empirical evidence from the e-MID (Italian electronic market for interbank deposits) dataset. In particular, we match several stylized facts observed in this market on a *daily* frequency, such as, in particular, the connectivity of the network, its low density, and near absence of intermediation.

The Global Financial Crisis has spurred the debates about the role of interconnectedness of a modern financial system in forming and shaping the systemic risk. Financial institutions are linked in many different ways and these linkages can lead to the propagation of shocks through the system. Recent theoretical literature on financial contagion models this process of shock propagation through various channels. In Elliott et al. (2014) the financial organizations are linked via cross-ownership of equity shares and the cascade of defaults is triggered by discontinuous loss of firm's value falling below a threshold; in Cabrales et al. (2014) different organizations are linked by investing to the same projects (i.e., by exchanging their assets) and the default of a project leads to the loss of value of several firms; in Glasserman and Young (2015) and Acemoglu et al. (2015) the financial institutions are linked through the interbank lending, so that a shortfall in assets of one bank can generate a domino effect due to not complete repayment on banks' liabilities. The common thread of all these models is that an initial shock to one bank affects its position causing it to change its balance sheet and this change will affect the balance sheets of other banks due to one or another existing connection.

The architecture of the network of linkages can play an important role for the shock propagation affecting both its size and the incurred losses. For instance, the roles of connectivity and density of the network¹ in cascades is often analyzed. Pioneering work of Allen and Gale (2000) indicated that the most systemically stable networks are fully connected, because in these networks the liquidity shocks can most easily be absorbed. However, recent literature on the shock propagation cited above adds a caveat that the relationship between connectivity and stability is not as simple. For instance, if a potential shock to the individual node is so large that the liquidity of the whole system cannot absorb it, the networks which

¹The network is *connected* if any node can be reached from any other node by following the links, in particular, directed links. If it is not connected, one can split the network on disjoint connected components and characterise the degree of connectivity on the basis of this split. On the other hand, within each connected component, one can look at the proportion of existing links with respect to all possible links. This proportion characterises *density* of the component. Component is called *fully connected*, if it has the largest possible density, i.e., all possible links are present.

are *less* connected will be more stable in that more nodes would survive such a shock.

Theoretical work is often limited to simple stylized networks, which are quite different from those that are observed in reality. At the same time, it is clear that the extent of instability and cascades is heavily affected the exact network topology. Several recent empirical studies suggested that the interbank lending network has a so-called *core-periphery* structure. Craig and Von Peter (2014) provide a definition of such a network as consisting of a core of the banks that lend and borrow from each other and the periphery banks that only borrow or lend from the core. They fit such a model to the large exposures data of banks located in Germany. van Lelyveld and in 't Veld (2014) use the large exposure data for the banks in the Netherlands and confirm that their structure is well described by the core-periphery model.

Importantly, any observed network is a result of decisions of profit-motivated financial entities.² An understanding of how the actual networks are formed and further analysis of the resulting structure is very important, especially if one wants to predict the consequences of shocks. Moreover, as Haldane and May (2011) point out, most important channels of shock propagation may not be those that involve balance sheet connections (as in the financial contagion models cited above) but those that affect the process of creation or destruction of such connections. With such indirect channels, the shocks will be amplified via drops in market liquidity either due to a general price fall for some classes of assets or because of a higher expectations of counter-party defaults (Gai and Kapadia, 2010a). Furthermore, shocks may cause a diminishing availability of the interbank loans, as the banks are hoarding on liquidity (Gai and Kapadia, 2010b).

Our work is also related to growing literature on trading in networks, where links represent transactions between nodes that are buyers and sellers. Because this market is decentralised by definition, some non-Walrasian mechanism of clearing is imposed.³ In the early literature, the roles of agents in trade was exogenous. Recently, Blume et al. (2009) and Nava (2015) endogenized trading roles.

In this paper we build a model of endogenous lending/borrowing decisions, which induce a network. Specifically, we model the overnight interbank market using the standard setup introduced in Poole (1968). In the model banks have to maintain the minimum reserve requirements. In the beginning of the period they can trade in a competitive interbank

²Endogenous formation of networks of financial institutions has been studied in the working paper version of Acemoglu et al. (2015), in the paper by Hommes et al. (2014) and by Babus and Hu (2015).

³For instance, in Kranton and Minehart (2001) an ascending auction is run between sellers to distribute the good in the network; in Corominas-Bosch (2004) and Manea (2011) buyers and sellers are involved in a repeated bargaining game on the network.

market to adjust their position, but they face uncertainty due to late settlements. The Central Bank in the model operates a corridor system, in which banks have access to the borrowing and lending facilities from the Central Bank. We extend the Poole (1968) model, by introducing the counter-party risk and considering decentralized, bilateral exchange. We focus on the question of which type of trading network can emerge under this setup and rely on the ideas from the network formation literature. Other recent contributions in a similar setup include Afonso and Lagos (2015) and Bech and Monnet (2013) who model decentralized exchange as we do, but use the search and matching framework and do not consider the network setting.

Our focus is on those networks that are aligned with banks' objective functions, i.e., the networks that are stable with respect to those deviations that increase individual payoff of banks. In particular, we extend the notion of *pairwise stability* of Jackson and Wolinsky (1996) (the seminal work for the literature on strategic network formation) to a context in which a binary decision to form a link, which represents a loan in our setting, is made jointly a direction of the loan, its amount and its interest rate.

We find that an extended concept of pair-wise stability produces a reach set of implications for the resulting networks. In particular, we are able to eliminate cycles and intermediaries in lending/borrowing. In equilibrium, we obtain a bipartite network in which borrowers and lenders form generically a unique component. Using comparative statics we find that an increase in the Central Bank borrowing or lending rates or minimum reserve requirements leads to an increase in the average interbank market rate. We also show that large increases in default probability may lead to "market freeze" where no trade is feasible. We find that our theoretical model reflects certain features of the actual overnight interbank market, e-MID.

The rest of this paper is organised as follows. In Section 2 we introduce the theoretical model of interbank lending decisions. The resulting endogenous decisions will induce a network of borrowing and lending. Sections 3 and 4 adapt the established equilibrium concept of pairwise stability to our context. This allows us to limit our attention to certain networks in equilibrium and to characterize them in terms of financial quantities and of network topology. Section 5 further refines our notion of pairwise stable equilibrium. Section 6 discusses the implications of monetary policy and the effects market conditions for our model. Section 7 provide some examples. Section 8 compares the predictions of our theoretical model with the empirical eMID data. Section 9 concludes.

2 Model

Consider a finite set \mathcal{N} of banks and the Central Bank. The Central Bank establishes the minimum reserve requirements on a fraction of banks deposits at the end of the day. We are not modeling deposits explicitly and simply denote the resulting requirement on the *reserve holdings* of bank $i \in \mathcal{N}$ by T_i . If reserve holdings at the end of the day are lower than T_i , bank i must borrow the difference at a *penalty rate* r^p . If, instead, reserve holdings are in excess of the required level, bank deposits the excess reserves to the Central Bank at a *deposit rate* r^d . The interest rates r^p and r^d with $r^p > r^d$ efficiently establish the upper and lower bounds for the interbank interest rates and are called ceiling and floor in the corridor system.

We model one day of banks operations as follows. In the beginning of the day, every bank $i \in \mathcal{N}$ predicts the net cash position from operations by the end of the day, s_i . In other words, s_i is the projected reserve holdings at the end of the day in the absence of borrowing or lending from the overnight market. The banks cannot perfectly predict their reserve holdings, however. It means that the *actual* reserve holdings (net of adjustments from the interbank market) are $s_i + \varepsilon_i$, where ε_i is a realisation of the random variable whose distribution is known to the bank. During the day, the banks have an access to the decentralized interbank market for overnight funds, where they are able to lend or borrow from each other in order to optimize their expected profit by adjusting their projected position of reserves s_i . Let C_i denote this adjusted position, i.e., the expected level of reserve holdings after the lending/borrowing in the interbank market. Finally, at the end of the day, after the interbank market is closed, the uncertainty about the actual level of reserve holdings is realized and the bank either deposit excess funds to the Central Bank or borrows from it to cover the minimum reserve requirements.

Our main focus of analysis will be on the interbank market for overnight funds. In this market, every pair of banks may enter into the bilateral agreement, when, say, bank i agrees to lend to bank j a quantity $\ell_{ij} > 0$ with interest rate r_{ij} . We do not rule out *a priori* that i and j may have two agreements, when i lends to j and, at the same time, borrows from j .⁴ We introduce the *lending matrix* $\mathbf{L} = \{\ell_{ij}\}_{i,j=1}^N$ with $\ell_{ij} \in \mathbb{R}_+$. (We set $\ell_{ii} = 0$ for all i .) This lending matrix induces the weighted directed network. There is a counter-party risk in the overnight market, i.e., with a small probability a borrower may default on its loan. To focus exclusively on the overnight market we do not model the origins of defaults, assuming that these origins are outside of the interbank market. Specifically, we make the following

⁴We prove later that this is not possible in equilibrium.

assumption.

ASSUMPTION 1. *Bank i has a probability of default, $q_i \in [0, 1]$, which is independent of banks' reserves and is known to every bank in the market. Defaults of different banks are independent events.*

Given the borrowing-lending network, every bank's projected reserves towards the end of the day is

$$C_i = s_i - \underbrace{\sum_{k \in \mathcal{N}} \ell_{ik}}_{\text{Lending}} + \underbrace{\sum_{m \in \mathcal{N}} \ell_{mi}}_{\text{Borrowing}}. \quad (1)$$

The first sum is over all borrowers of bank i and the second sum is over all lenders to bank i . After the interbank market is closed, the uncertainty about the actual banks reserve holdings is resolved. The reserve holdings of bank i are given by

$$(s_i + \varepsilon_i) - \sum_{k \in \mathcal{N}} \ell_{ik} + \sum_{m \in \mathcal{N}} \ell_{mi} = C_i + \varepsilon_i$$

where ε_i is a realization of a continuous random variable with CDF function F_i and density f_i . Thus, $s_i + \varepsilon_i$ is the actual net cash position from usual banking operations, and ε_i can be interpreted as an error made in predicting this variable in the beginning of the day. We assume that the distribution F_i is known and its mean is 0. The expected payoff on the overnight position of bank i in the Central Bank is

$$\pi_i^{\text{CB}} = - \int_{-\infty}^{T_i - C_i} (T_i - (C_i + \varepsilon_i)) r^p dF_i(\varepsilon_i) + \int_{T_i - C_i}^{\infty} ((C_i + \varepsilon_i) - T_i) r^d dF_i(\varepsilon_i). \quad (2)$$

The first term gives an expected loss that the bank incurs by not meeting the reserve requirements and thus the integration is over the cases when $T_i > C_i + \varepsilon_i$. The second term integrates over the remaining instances and represents an expected gain of the bank for having excess reserve holdings.

Banks enter the interbank market in order to adjust their reserve holdings and thus affect their expected profits π^{CB} given by (2). In this market, borrowing banks bear interest expenses, whereas lending banks earn interest, but are subject to the counter-party default risk. In accordance with Assumption 1 the bank's default probability is determined by its overall standing and is not affected by simple overnight market operations. The next Assumption discusses the consequences of default for the interbank market.

ASSUMPTION 2. *If a borrower defaults, then the lender will not receive its loan (neither principal nor interest) back. If a lender defaults, then the borrower will have to pay back both*

principal and interest of a loan to some centralized institution.

Under Assumption 2 the expected profit from the transactions on the interbank market of bank i is

$$\pi_i^{\text{IM}} = \underbrace{\sum_{k \in \mathcal{N}} r_{ik} \ell_{ik} (1 - q_k)}_{\text{interest due from repaying borrowers}} - \underbrace{\sum_{k \in \mathcal{N}} \ell_{ik} q_k}_{\text{losses because of defaulted borrowers}} - \underbrace{\sum_{m \in \mathcal{N}} \ell_{mi} r_{mi}}_{\text{interest due to lenders}} \quad (3)$$

Note an asymmetry in the profit which is a consequence of Assumption 2.⁵ The first two terms represent the expected payoff from the bank's borrowers: if a borrower honors the loan, bank i gains in the interest paid on the loan, and if a borrower defaults, bank i loses its loan. The third term is the interest payment on own loans which bank i should make.

The total expected profit that bank i derives from the overnight market is then

$$\pi_i = \pi_i^{\text{CB}} + \pi_i^{\text{IM}}, \quad (4)$$

where the two terms are given by (2) and (3). Bank i receives this profit only when it does not default (i.e., with probability $1 - q_i$). As default is an exogenous event to the overnight market operations, without loss of generality⁶ we assume that the objective of bank i is to maximize π_i in (4).

As it is mentioned above, banks have a complete knowledge about probabilities of defaults of counter-party in the interbank market but are subject to uncertainty regarding final reserve holdings. In this environment, each of the banks is willing to maximise (4). This leads to the following trade-offs for borrowers and lenders. When bank i borrows some funds in the interbank market, it increases its own probability to meet the reserve requirements (and overall increases π^{CB}) but has to pay an interest on the loans (thus decreasing π^{IM}). Note that for the borrower the counter-party does not matter in this trade-off, as it should pay an interest anyway according to Assumption 2. For lenders the trade-off is more complicated. When bank i lends funds, it decreases its own probability to meet the reserve requirements (and the whole expression in (2)) but may increase or decrease the net cash flow from the borrowers (the first two terms in (3)) depending on the interest rate and the probability of counter-party default. This also means that for a lender the choice of counter-party matters.

⁵The assumption that if a borrower defaults, then the lender will not receive its cash (neither principal nor interest) back is made for simplicity. It can be easily relaxed by assuming that a defaulted borrower repays a fraction $\gamma \in [0, 1]$ of its obligations (so-called "hair-cut"). Then, expression (3) will still hold with q_k substituted by $(1 - \gamma)q_k$. An analogous version of the same assumption is also in the recent paper by Blasques et al. (2015).

⁶Let B_i be an exogenous payoff in the case if bank i defaults. Assumption 1 guarantees that maximization of $(1 - q_i)\pi_i + q_i B_i$ is equivalent to maximization of π_i .

3 Pairwise Stable Equilibrium Networks

Given exogenous variables, i.e., penalty rate r^p , deposit rate r^d , predicted net cash positions of different banks s_i , reserve requirements T_i , the CDF function F_i for the funds' uncertainty shock ε_i , and the banks probabilities of default q_i , the banks enter the interbank lending market and form bilateral agreements. The outcome is summarized by the lending matrix $\mathbf{L} = \{\ell_{ij}\}$ and by the matrix of the corresponding interest rates $\mathbf{r} = \{r_{ij}\}$ assigned to the positive elements of matrix \mathbf{L} . This structure induces the directed weighted network that we denote $g = (\mathbf{L}, \mathbf{r})$. Whenever $\ell_{ij} > 0$, we say that there is a directed link from i to j and that this link has two weights, positive amount ℓ_{ij} that bank i lends to bank j and the interest rate $r_{ij} \geq 0$ corresponding to this loan.

For a given network g we can calculate, using (1), the projected reserves of every bank towards the end of the day but before the uncertainty is resolved. We denote these reserves $C_i(g)$ and call them the *interim* reserves of bank i . Using (2), (3), and (4), we can compute the expected payoff of bank i , which is denoted as $\pi_i(g)$.

We will limit our attention to the networks that are consistent with banks' incentives to maximise their payoffs (4) taking into account existing network structure. In this sense, we consider *strategically stable* networks. Inspired by a milestone notion introduced by Jackson and Wolinsky (1996), we consider networks that are stable in the sense of the following definition.

DEFINITION 1. *Network $g = (\mathbf{L}, \mathbf{r})$ is a **pairwise stable equilibrium** if the following conditions are met:*

1. *no single bank i can remove an existing link in which it is involved (i.e., for any $k \in \mathcal{N}$ with $\ell_{ik} > 0$ and for any $m \in \mathcal{N}$ with $\ell_{mi} > 0$) and be better off;*
2. *no couple of banks i and j with link $\ell_{ij} > 0$, can change their agreement to $(\tilde{\ell}_{ij}, \tilde{r}_{ij})$ (keeping the rest of the network the same) that would make bank i better off without making bank j worse off;*
3. *no couple of banks i and j with $\ell_{ij} = 0$, can find an agreement $(\tilde{\ell}_{ij}, \tilde{r}_{ij})$ with $\tilde{\ell}_{ij} > 0$ that would make both banks i and j better off.*

The first requirement says that in the pairwise stable equilibrium network, any two linked banks find their deal profitable, that is none of them would prefer to remove the link unilaterally. The second requirement further specifies what type of the deal the banks are agree upon. Namely, it says that any bilateral agreement, in the equilibrium, is Pareto

efficient for the two involved banks given the rest of the network. The third requirement says that banks have no incentive to add a new link to the network.

Comparing with a standard notion of pairwise stability as introduced in Jackson and Wolinsky (1996), our concept is an adaptation of their notion to a setting where every link have *three* payoff-relevant variables: the direction of the loan, the loan amount and the interest rate. We assume that if a potential link between i and j may be created, the network configuration is not in equilibrium. Then, the process of link creation is interpreted as a bargaining process between banks i and j . The second requirement of Definition 1 provides some discipline on the outcome of this bargaining without making more specific assumptions about banks' behavior. Finally, as Jackson and Wolinsky (1996), we assume that every bank can unilaterally cut any existing link.

In the rest of this section we will characterize topological properties of network $g = (\mathbf{L}, \mathbf{r})$ that are consistent with pairwise stable equilibrium.

3.1 Payoff improving links and Feasibility sets

Let us begin by studying the incentives that the two individual banks have to create or sustain a link. For this purpose we introduce the following notation. For a given network g and for a pair of banks i and j with $\ell_{ij} > 0$, we denote as ij the link from i to j , and as $g - ij$ the network obtained from g by deleting this link. In the opposite situation when $\ell_{ij} = 0$ in network g , we denote $g + ij$ a network obtained from g by adding a link along which i lends to j some fixed and positive amount ℓ with some interest rate r . For the latter case, when a link between i and j is added, we introduce the quantities

$$\Delta_i^{g+ij}(\ell, r) = \pi_i(g + ij) - \pi_i(g),$$

and

$$\Delta_j^{g+ij}(\ell, r) = \pi_j(g + ij) - \pi_j(g),$$

that represent the changes in the expected profit of the lender and borrower, respectively, in the case if they would agree on link (ℓ, r) . Furthermore, let us introduce the *feasibility regions*, $F_{i \rightarrow j}^L(g)$ for bank i as for a lender to j , and $F_{j \leftarrow i}^B(g)$ for bank j as for a borrower from i , as sets

$$F_{i \rightarrow j}^L(g) = \{(\ell, r) : \ell > 0, r \geq 0, \Delta_i^{g+ij}(\ell, r) > 0\},$$

and

$$F_{j \leftarrow i}^B(g) = \{(\ell, r) : \ell > 0, r \geq 0, \Delta_j^{g+ij}(\ell, r) > 0\}.$$

These sets contain all pairs of ℓ and r which make lender i strictly better off from a link with j and borrower j strictly better off from a link with i , respectively.

Note that for the network g' , which is the same as g except that link (ℓ_{ij}, r_{ij}) is present, we can write the changes in the expected profit of the lender and borrower when this link is removed as

$$\pi_i(g') - \pi_i(g' - ij) = \Delta_i^{g+ij}(\ell_{ij}, r_{ij}), \quad \text{and} \quad \pi_j(g') - \pi_j(g' - ij) = \Delta_j^{g+ij}(\ell_{ij}, r_{ij}).$$

The notation is consistent with the previous one because the rest of the network is held the same.

With this notation, conditions 1 and 3 of Definition 1 can be reformulated as follows. The first condition requires for a pairwise stable network g' , where link ij (with $\ell_{ij} > 0$ and $r_{ij} \geq 0$) is present, that for $g = g' - ij$ it is

$$\Delta_i^{g+ij}(\ell_{ij}, r_{ij}) \geq 0 \quad \text{and} \quad \Delta_j^{g+ij}(\ell_{ij}, r_{ij}) \geq 0.$$

Jointly these two inequalities mean that the pair (ℓ_{ij}, r_{ij}) lies in the intersection of the closures of feasibility regions, $\text{Cl } F_{i \rightarrow j}^L(g) \cap \text{Cl } F_{j \leftarrow i}^B(g)$, for the network $g = g' - ij$. The third condition requires for a pairwise stable network g with non-existing link $\ell_{ij} = 0$ to have no pair of (ℓ, r) such that

$$\Delta_i^{g+ij}(\ell, r) > 0 \quad \text{and} \quad \Delta_j^{g+ij}(\ell, r) > 0.$$

This means that the intersection of feasibility regions must be empty $F_{i \rightarrow j}^L(g) \cap F_{j \leftarrow i}^B(g) = \emptyset$.

Our next natural step is to analyze how the feasibility sets of lender i and borrower j look like for an arbitrary pair of banks in a given network, conditional on the rest of the network. The following quantity will play an important role in the analysis. For given network g and every bank i define

$$W_i = r^d + (r^p - r^d) F_i(T_i - C_i) = r^p F_i(T_i - C_i) + r^d (1 - F_i(T_i - C_i)). \quad (5)$$

This is the *expected marginal rate* of bank i 's transaction with the Central Bank. When bank i decides to lend an extra reserves to any other bank in the interbank market, the profit of i from the Central Bank interactions will expect to drop marginally by W_i . Analogously, when bank i decides to borrow extra reserves from any other bank in the interbank market, the profit of i from the Central Bank interactions will expect to increase marginally by W_i .

The following lemma summarizes some simple but useful properties of the expected marginal rate of a bank in the network.

LEMMA 1. *Consider an arbitrary (not necessarily equilibrium) network g and bank i with reserve requirements T_i . Then:*

1. $r^d \leq W_i \leq r^p$.
2. W_i is a decreasing function of the interim reserves C_i . In particular, it increases with extra lending that i makes and decreases with extra borrowing it makes.
3. $W_i \rightarrow r^p$ when $C_i \rightarrow -\infty$ and $W_i \rightarrow r^d$ when $C_i \rightarrow +\infty$.

Now let us take two banks i and j from network g and fix the rest of the network. We analyze all possible loans (ℓ, r) that i may offer to j , i.e., any $\ell \geq 0$. Note that both W_i and W_j depend on ℓ but do not depend on r . We will write this dependence explicitly as $W_i(\ell)$ and $W_j(\ell)$. Lemma 1 implies that $W_i'(\ell) > 0$ and $W_j'(\ell) < 0$.

We characterize the feasibility sets for a loan between lender i and borrower j as follows.

LEMMA 2. *Consider an arbitrary network g and banks i and j . Let bank i consider to lend reserves to bank j , and let the rest of the network be fixed. Then:*

1. *feasibility set of the lender is the strict epigraph of function $h^L : \mathbb{R}^+ \rightarrow \mathbb{R}$, i.e.,*

$$F_{i \rightarrow j}^L(g) = \{(\ell, r) : \ell > 0, r > h^L(\ell)\} .$$

Moreover, function h^L is strictly increasing and

$$h^L(0) = \frac{q_j + W_i(0)}{1 - q_j} . \tag{6}$$

2. *feasibility set of the borrower is the strict hypograph of function $h^B : \mathbb{R}^+ \rightarrow \mathbb{R}$, i.e.,*

$$F_{j \leftarrow i}^B(g) = \{(\ell, r) : \ell > 0, r < h^B(\ell)\} .$$

Moreover, function h^B is strictly decreasing and

$$h^B(0) = W_j(0) . \tag{7}$$

Proof. We use the result of Lemma 1 and the derivatives of the payoff functions of lender and borrower as computed in Appendix A. Note that the derivatives of function $\Delta_i^{g+ij}(\ell, r)$ coincide with the derivatives of $\pi_i(\ell, r)$ when i lends to j funds ℓ with interest rate r , and so $\partial\Delta_i^{g+ij}/\partial\ell = r(1 - q_j) - q_j - W_i(\ell)$ and $\partial\Delta_i^{g+ij}/\partial r = 1 - q_j$. Similarly, the derivatives of function $\Delta_j^{g+ij}(\ell, r)$ coincide with the derivatives of $\pi_j(\ell, r)$ when j is a borrower of funds ℓ with interest rate r from i , and so $\partial\Delta_j^{g+ij}/\partial\ell = -r + W_j(\ell)$ and $\partial\Delta_j^{g+ij}/\partial r = -\ell$.

Consider lender i and start with the boundary of the feasibility set where $\ell = 0$. On this boundary $\Delta_i^{g+ij}(\ell, r) \equiv 0$ and $\partial\Delta_i^{g+ij}/\partial\ell = r(1 - q_j) - q_j - W_j(0)$. This derivative is positive for $r > r^*$, where threshold r^* is defined by the right hand-side of (6); and it is negative for $r < r^*$. Consider the case, when $r < r^*$. Then for any $\ell > 0$ we have $\partial\Delta_i^{g+ij}/\partial\ell = r(1 - q_j) - q_j - W_j(\ell) < r(1 - q_j) - q_j - W_j(0) < 0$. In other words, function Δ_i^{g+ij} starts with value 0 and strictly decreases with ℓ in this case. Therefore, all points (ℓ, r) with $r < r^*$ are outside of the feasibility set.

Let us now fix arbitrary $\tilde{\ell} > 0$. We have established that when r is sufficiently small, $\Delta_i^{g+ij}(\tilde{\ell}, r) < 0$. But Δ_i^{g+ij} is a strictly increasing function of r , converging to $+\infty$ when $r \rightarrow \infty$, according to (3) (and because (2) does not depend on r). Therefore, there exists a unique value of $\tilde{r} > 0$ for which $\Delta_i^{g+ij}(\tilde{\ell}, \tilde{r}) = 0$. Point $(\tilde{\ell}, \tilde{r})$ thus belongs to the boundary of the feasibility set, whereas all the points $(\tilde{\ell}, r)$ with $r > \tilde{r}$ in the feasibility set. This proves that the feasibility set of a lender is the strict epigraph of function h^L defined for $\tilde{\ell} > 0$ by mapping $\tilde{\ell} \mapsto \tilde{r}$. Moreover, as we showed above $h^L(0) = r^*$ and is given by (6).

To show that function h^L is increasing, fix some $r > h^L(0)$. We have established that for this r and for sufficiently small ℓ , points (ℓ, r) are in the feasibility region. Moreover, function $\Delta_i^{g+ij}(\ell, r)$ as a function of ℓ is concave, since $\partial^2\Delta_i^{g+ij}/\partial\ell^2 = -W_i'(\ell) < 0$. It means that for any $r > h^L(0)$ there is at most one point (ℓ, r) where $\Delta_i^{g+ij}(\ell, r) = 0$. This implies that function h^L is strictly increasing. If it would not be, then it would be possible to find some $\ell_1 < \ell_2$ with $h^L(\ell_1) \geq h^L(\ell_2) > h^L(0)$. But then, for any $\hat{r} \in [h^L(\ell_2), h^L(\ell_1)]$ there would be at least two values of ℓ where $\Delta_i^{g+ij}(\ell, \hat{r}) = 0$.

The proof of the second statement of this lemma, for a borrower, is similar. \square

The results of Lemma 2 are illustrated in the left panel of Fig. 1. In this example, whose parameters and other details will be discussed in Section 7, we depict, in the coordinates (ℓ, r) the feasibility sets of a lender (light red) and of a borrower (light blue). The higher (blue) dot marks the highest interest rate which a borrower can offer, i.e., $h^B(0)$ as given by (7). The lower (red) dot marks the lowest interest rate at which a lender can accept the deal, i.e., $h^L(0)$ as given by (6). The two horizontal lines show the penalty rate r^p and the deposit rate r^d of the Central Bank.

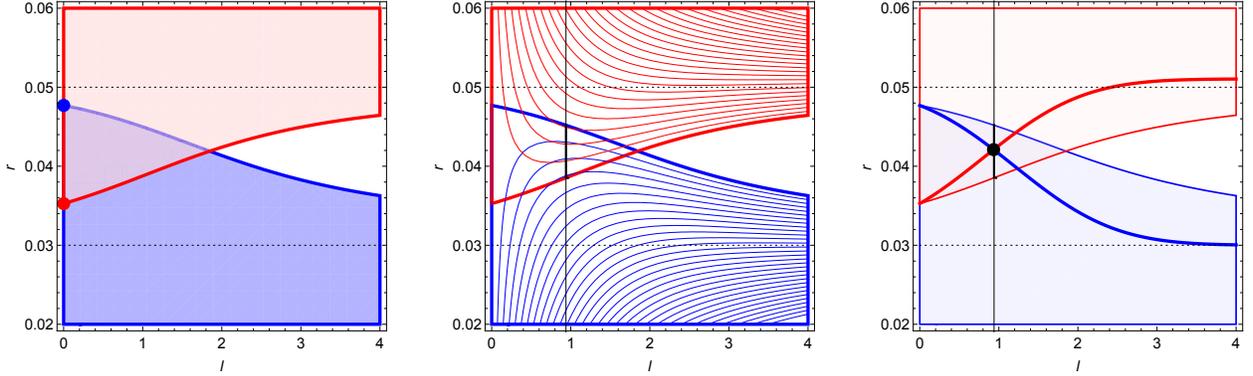


Figure 1: Illustration of Example with two banks from Section 7. *Left:* The feasibility sets of a lender (red) and of a borrower (blue). *Middle:* Indifference curves of lender and borrower and the contract curve (thick solid vertical line). *Right:* Supply (thick red) and demand (thick blue) curves under competitive behavior and the unique competitive pairwise stable equilibrium (black dot).

In this example, the highest interest rate of the borrower is within the corridor $[r^d, r^p]$ set by the Central Bank. Inspecting (7) we conclude that this is a general property. In this example also the lowest interest rate of the lender is within the corridor, but this is not general. Indeed, from (6) it follows that this interest rate is not less than r^d , but can be higher than r^p , when the counter-party has a high default probability.

Note also that in the example the feasibility regions of lender and borrower intersect, suggesting that these two banks should have a link in a pairwise stable equilibrium. From the properties of the boundaries of the feasibility sets as described in Lemma 2, it follows that the feasibility region will not be empty as soon as $h^B(0) > h^L(0)$. Then we obtain the following result.

PROPOSITION 1. *Consider two banks i and j with $l_{ij} = 0$ in the network. Loan (ℓ, r) from i to j that makes both banks better off exists if and only if*

$$q_j + W_i(0) < (1 - q_j)W_j(0). \quad (8)$$

When $q_j = 0$, condition (8) becomes simply $W_i(0) < W_j(0)$ which is equivalent to $F_i(T_i - C_i) < F_j(T_j - C_j)$. For identical distributions F_i and F_j and equal reserve requirements $T_i = T_j$, this implies that the link from i to j will be created if and only if $C_i > C_j$. In case if $C_i > C_j$, introducing counter-party risk, $q_j > 0$, will increase the lower bound $h^L(0)$ for

the lender, so that the intersection of feasibility sets will be shrinking.⁷ This intersection is getting larger with a higher degree of asymmetry in the initial position of the banks. An asymmetry in reserve requirements or in the parameters of F (e.g., different variance of the shock to the predicted reserves) will also affect the region of all (ℓ, r) that improve payoffs of both banks. Higher reserve requirements will lead to higher borrowing incentives *ceteris paribus* and higher variance of the reserve shock will enhance the asymmetry.

In light of Definition 1, condition (8) has two important implications for the pairwise stable equilibrium networks. First, whenever the link from i to j exists in such network, then in the network without this link this inequality must be satisfied. Second, if the link from i to j does not exist in such network, it means that the intersection of the feasibility regions $F_{i \rightarrow j}^L$ and $F_{j \leftarrow i}^B$ is empty and, hence, the inverse inequality holds. We formulate it as

COROLLARY 1. *In every pairwise stable equilibrium network, if link from i to j does not exist then $q_j + W_i \geq (1 - q_j)W_j$.*

3.2 Contract Curve

For the two banks i and j which can individually gain from an ij link, the problem of finding a loan contract (ℓ_{ij}, r_{ij}) between them can be described as a bargaining problem within their feasibility set given by the intersection of $F_{i \rightarrow j}^L$ with $F_{j \leftarrow i}^L$. The second requirement of Definition 1 imposes that in the pairwise stable equilibria, the banks should find themselves in the point which is Pareto-efficient for them given the rest of the network. In other words, whenever $\ell_{ij} > 0$ in network g , the banks i and j must choose (ℓ_{ij}, r_{ij}) in such a way that no Pareto-efficient deviation from this point is possible. The set of such points within the feasibility set is called *contract curve*.

The contract curve in the coordinates (ℓ_{ij}, r_{ij}) is characterized by the condition that the slopes of indifference curves of the two banks coincide. Therefore it holds that

$$-\frac{\partial \pi_i / \partial \pi_i}{\partial \ell_{ij} / \partial r_{ij}} = -\frac{\partial \pi_j / \partial \pi_j}{\partial \ell_{ij} / \partial r_{ij}}. \quad (9)$$

With a bit of computations (see Appendix A) we find that this condition becomes

$$q_j + r^d + (r^p - r^d)F_i(T_i - C_i) = (1 - q_j) \left(r^d + (r^p - r^d)F_j(T_j - C_j) \right),$$

⁷Note that if $W_i(0) \geq W_j(0)$ (which for identical distributions and reserve requirements is equivalent to $C_i \leq C_j$), then also $q_j + W_i(0) \geq (1 - q_j)W_j(0)$, and, hence, the link will not be created *irrespective* of the level of counter-party risk.

or, using the expected marginal rate notation introduced in (5), simply

$$q_j + W_i = (1 - q_j)W_j. \quad (10)$$

This condition, that should be satisfied in the pairwise stable equilibrium for every existing link, relates the quantities W_i and W_j for every two pair of lender and borrower. Note that the expected marginal rates, W 's, do not depend on interest rate r_{ij} . They both depend on ℓ_{ij} , and the equation (10), hence, can be thought of as an equation to determine ℓ_{ij} given the rest of the network. Next result assures that if such ℓ_{ij} exists, then, it is unique, and moreover established a necessary and sufficient condition for an existence of (ℓ_{ij}, r_{ij}) on the contract curve.

PROPOSITION 2. *Consider an arbitrary network g , and any two banks i and j with non-empty intersection of feasibility sets $F_{i \rightarrow j}^L$ with $F_{j \leftarrow i}^B$ and such that $ij \notin g$. Then there exists a unique $\ell_{ij} > 0$ that is consistent with requirement 2 from Definition 1 of the pairwise stable equilibrium.*

Proof. When an intersection of the feasibility sets is not empty, Proposition 1 implies that $q_j + W_i(0) < (1 - q_j)W_j(0)$. When ℓ increases, the left hand-side strictly increases and the right hand-side strictly decreases. Lemma 1 implies that when $\ell \rightarrow \infty$, the left hand-side approaches $q_j + r^p$, whereas the right hand-side approaches $(1 - q_j)r^d$. Since $q_j + r^p \geq r^p > r^d > (1 - q_j)r^d$, there exists a unique $\ell_{ij} > 0$ that solves equation (10).

We now have to show that for this ℓ_{ij} there is at least one interest rate r such that (ℓ_{ij}, r) are in the intersection of the feasibility sets. Consider $r_{ij} = (q_j + W_i)/(1 - q_j) = W_j$, where the last equality is due to (10). It is easy to check that in the point (ℓ_{ij}, r_{ij}) the indifference curves of both lender and borrower are horizontal. From Lemma 2 we know that the indifference curve corresponding to $\Delta_i^{g+ij}(\ell, r) = 0$ is given by function $h^L(\ell)$ and so its slope in ℓ_{ij} is positive. It is easy to see that the slope of indifference curve of lender for a given ℓ decreases with r . Therefore, $r_{ij} > h^L(\ell_{ij})$, which means that $(\ell_{ij}, r_{ij}) \in F_{i \rightarrow j}^L$. In a similar way, we can show that $(\ell_{ij}, r_{ij}) \in F_{j \leftarrow i}^B$. \square

The central panel of Fig. 1 illustrates the previous result. We show there several indifference curves of both lender (red) and borrower (blue) belonging to their feasibility sets (i.e., these are isoprofit curves when profits are larger than in the empty network). The vertical black line represents all the points in coordinates (ℓ, r) for which Eq. (9) is satisfied. In this example, the intersection of the feasibility sets is not empty and so the contract curve exists as shown by the thick vertical line. There is a whole interval of interest rates that are consistent with the pairwise stable equilibrium network.

Proposition 2 immediately implies the following

COROLLARY 2. *In every pairwise stable network, for any two banks with i and j with a link $\ell_{ij} > 0$, it holds that $q_j + W_i = (1 - q_j)W_j$.*

4 Equilibrium Network Configurations

We now apply the results from the previous section to characterize the networks that can emerge in the pairwise stable equilibrium. As the discussion above implies, there might be infinitely many such equilibria, because requirements of Definition 1 are too weak to fix the interest rate. Some extra assumptions are needed and we will discuss them in the next section. On the other hand, as we will show now, Definition 1 is sufficient not only to give quite precise predictions about the loan amounts, given the values of the exogenous variables, but also describe which types of networks should in general be observed in the pairwise stable equilibrium.

Our first result shows that a pairwise stable equilibria cannot have pair of banks that lend to each other.

LEMMA 3. *In a pairwise stable equilibrium, if $\ell_{ij} > 0$ for a pair of banks, i and j , and if either $q_i > 0$ or $q_j > 0$, then $\ell_{ji} = 0$.*

Proof. Suppose the contrary, i.e., that i and j lend funds to each other. Consider bank i . As it is both lender and borrower for j , Corollary 2, implies that

$$W_i = (1 - q_j)W_j - q_j = (1 - q_j)(1 - q_i)W_i - (1 - q_j)q_i - q_j < (1 - q_j)(1 - q_i)W_i,$$

where in the last inequality we used the fact that at least one of the two banks have non-zero probability of default. Clearly the last inequality implies that $W_i < W_i$, which is a contradiction. \square

The second result shows that at every equilibrium, we cannot have directed loops.

LEMMA 4. *In a pairwise stable equilibrium, if we have a finite set of banks $\{i_1, i_2, \dots, i_k\}$, with $q_j > 0$ for at least one $j \in \{i_1, i_2, \dots, i_k\}$, such that $\ell_{i_h i_{h+1}} > 0$ for any consecutive pair of banks, then $\ell_{i_k i_1} = 0$.*

Proof. From Corollary 2 we have that in such pairwise stable equilibrium network for any $1 \leq h \leq k$ it is

$$W_{i_h} = (1 - q_{i_{h+1}})W_{i_{h+1}} - q_{i_{h+1}} \leq W_{i_{h+1}}.$$

Suppose the contrary, i.e., that $\ell_{i_k i_1} > 0$ in the equilibrium. Then $W_{i_k} \leq W_{i_1}$ and generally

$$W_{i_k} \leq W_{i_1} \leq W_{i_2} \leq \dots \leq W_{i_k}$$

with at least one strict inequality. But this is impossible. \square

The third result shows that there are no directed paths of length greater than one or, in other words, there is no intermediary.

LEMMA 5. *In a pairwise stable equilibrium, we cannot have three banks $\{i, j, k\}$, with $q_j > 0$, such that $\ell_{ij} > 0$ and $\ell_{jk} > 0$.*

Proof. Assume the contrary, i.e., there are three banks with $\ell_{ij} > 0$ and $\ell_{jk} > 0$. First, we will show that then it also must be the case that $\ell_{ik} > 0$. Indeed, from Corollary 2 we have that

$$W_i = (1 - q_j)W_j - q_j < W_j = (1 - q_k)W_k - q_k.$$

This means that $q_k + W_i < (1 - q_k)W_k$ and from Corollary 1 there must be a positive loan from i to k in the equilibrium.

But then from Corollary 2 we have that

$$W_i = (1 - q_k)W_k - q_k = W_j,$$

where the first equality is because i lends to k and the second equality is because j lends to k . Therefore, $W_i = W_j$, which is impossible given that i lends to j with $q_j > 0$. \square

The fourth result shows that there will generically be no separated components in the pairwise stable equilibrium network.

LEMMA 6. *In a pairwise stable equilibrium, we cannot have four banks $\{i, j, k, h\}$, such that $\ell_{ij} > 0$, $\ell_{kh} > 0$, and $W_i \neq W_k$.*

Proof. Suppose the contrary, that is for those four banks $\ell_{ij} > 0$, $\ell_{kh} > 0$, and, say, $W_i > W_k$. From Corollary 2, for the link from i to j we have that

$$(1 - q_j)W_j - q_j = W_i.$$

Therefore, $W_k < (1 - q_j)W_j - q_j$, and Corollary 1 implies that then it must be $\ell_{kj} > 0$. But then, along this link, applying Corollary 2, we have that $W_k = (1 - q_j)W_j - q_j$, which is a contradiction.

If $W_k > W_i$, then similar reasoning applies to the links to h . \square

All four lemmas of this section imply the following result.

PROPOSITION 3. *A pairwise stable equilibrium is such that banks are partitioned in three groups: isolated banks, borrowers and lenders. Borrowers and lenders form generically a unique component where all directed paths have at most length one.*

5 Interest rate under competitive behavior

Corollaries 1 and 2 give an analytical characterization for every existing and non-existing link in the pairwise stable networks. Condition (10), allows one to determine the loan amounts in a unique way for every existing link, given the rest of the network. At the same time, the notion of pairwise stable networks is too weak to specify interest rates precisely, though it imposes certain boundary conditions on them. The next notion is a possible refinement of the pairwise stable equilibria that leads to more strict predictions about the interest rates observed in the network.

DEFINITION 2. *Network $g = (\mathbf{L}, \mathbf{r})$ is a **competitive pairwise stable equilibrium** if it is the pairwise stable equilibrium and on every existing link ij , the amount $\ell_{ij} > 0$ maximises the profits of both lender i and borrower j given the interest rate r_{ij} .*

In a competitive pairwise stable equilibrium the banks exhibit the price-taking behavior in their bilateral interactions. This assumption may be realistic when the number of banks is large and when there is approximately as many potential lenders (banks expecting high s 's) and potential borrowers (banks expecting low s 's).

To study the network in the competitive pairwise stable equilibrium, let us assume, as before, that given the rest of the network, bank i considers lending some reserves to bank j . For lending bank i , differentiating π_i with respect to ℓ_{ij} , we find that the first order-condition is given by

$$r_{ij}(1 - q_j) = q_j + r^d + (r^p - r^d) F_i(T_i - C_i). \quad (11)$$

Moreover, the second-order condition guarantees that the solution of this equation delivers the maximum profit to a lender, given the interest rate r_{ij} .

When bank i decides to lend to bank j , it compares the marginal cost of this decision given by the right hand-side of (11) with the marginal benefit of it given by the left hand-side of (11). The marginal cost are equal to the probability of default of the borrower (as then the principal is lost) plus the marginal increase in the expected cost of dealing with the Central Bank. The marginal benefit is the expected marginal payment from bank j . Note

that the interest rate consistent with the lender's first-order condition is always above r^d but may also be above r^p . This is a consequence of a risk that bank j may default.

Note that (11) can be written as $r_{ij}(1 - q_j) = q_j + W_i(\ell_{ij})$. This implicitly defines the supply function of lender i for the funds to borrower j . Since $W_i' > 0$, the supply function is strictly increasing.

Consider now bank j who is borrowing from i . Differentiating π_j with respect to ℓ_{ij} , we find that the first order-condition for j is given by

$$r_{ij} = r^d + (r^p - r^d) F_j(T_j - C_j). \quad (12)$$

The second-order condition is again satisfied. When bank j decides to borrow, it compares the marginal cost of this decision given by the left hand-side of (12) with the marginal benefit of it given by the right hand side of (12). The marginal cost of borrowing is given by the interest rate r_{ij} . The marginal benefit is the marginal decrease in the expected cost of dealing with the Central Bank.

Note that the rate that borrower j pays does not depend on the identity of lender, but only on the lending amount, according to (12). Therefore, in the competitive pairwise stable equilibrium, there will be only one interest rate per borrower, which we call this borrower's interest rate and denote as r_j . Note also that this interest rate is always between r^d and r^p .

Rewriting (12) as $r_{ij} = W_j(\ell_{ij})$ defines implicitly the demand function for the borrower j . This function is decreasing as $W_j' < 0$.

Combining (11) and (12) we obtain that in the competitive pairwise stable equilibrium it must be that

$$r_j = \frac{q_j + W_i(\ell_{ij})}{1 - q_j} = W_j(\ell_{ij})$$

The second equality is equivalent to (10), confirming that competitive behavior is consistent with Pareto efficiency. In fact, this result shows that as soon as in the network the contract curve for banks i and j is not empty, there exists a unique pair of (ℓ_{ij}, r_{ij}) that satisfies the first order conditions of both lender and borrower. Moreover, as we showed in the proof of Proposition 2, this point (ℓ_{ij}, r_j) belongs to the feasibility regions of both buyer and seller. In this sense the notion of competitive pairwise stable equilibrium refines the pairwise stable equilibrium from Definition 1.

We illustrate the competitive pairwise stable equilibrium in the right panel of Fig. 1, where the thick red and blues curves show, respectively, the supply of loans and demand for loans schedules as derived from (11) and (12). The black point gives the equilibrium values of ℓ and r .

6 Monetary policy and market conditions

The Central Bank may influence the interbank market interest rate and volumes through a range of monetary policy instruments. In turn, banks pass through the interest rate changes to the rest of the economy. In our model the policy variables are: deposit rate r^d , penalty rate r^p , and bank-specific minimum reserve requirements T 's. Indirectly through open market operations (buying/selling government bonds, lending through repurchase agreements), the Central Bank may influence the overall liquidity and level of bank reserves in the economy.⁸ As we showed in Section 5, under competitive behavior the interest rate of the borrower is always between r^d and r^p , and the interest rate of the lender is above r^d , but may be above r^p . It is easy to see that r^d and r^p affect the marginal rate of banks transaction with the Central Bank, W 's as defined in (5). Hence, increasing r^d or r^p will increase W 's and the average overnight market interest rate, *ceteris paribus*. Intuitively increase in the Central Bank deposit rate r^d , will shift the lender's supply curve up as their outside option of depositing money with the bank will improve. On the other hand, an increase in r^p will result in an upward shift of the demand curve from the borrowers as they will face a larger penalty rate. Similarly, minimum reserve requirements T_i 's are positively related to W_i 's and their increase will result in higher average overnight market rate. Finally, open market operations target the overall liquidity in the system and thus affect s_i 's. Since W_i is negatively related with s_i , an increase of the overall liquidity will decrease the average overnight interest rate.

Market conditions may also have an effect on the overnight market rate and transaction volumes. When market conditions worsen, the probabilities of default, q_i , may increase. This will shift the supply curve of the lenders up as their expected marginal payoff will decrease (when the increased defaults are factored in). The demand curve of the borrowers will not be affected. This will result in a higher equilibrium overnight market rate and smaller number of transactions. For a sufficiently high default probability, the supply curve may shift above the demand curve for all r 's and result in "market freeze" with zero transaction volumes and undefined overnight market rate.

Interestingly, increased uncertainty about the reserves, σ , has different implications for lenders and borrowers. Specifically, an increase in σ results in an increase in W in the case when $T > C$ and in a decrease in W when $C < T$. This implies that the demand curve of

⁸The range of used policy instruments differs from country to country. For instance, the ECB employs all mentioned instruments. Until 6 October 2008, the US Fed paid zero deposit rate on excess reserves and introduced positive deposit rate as an anti-crisis measure. The People's Bank of China actively uses the minimum reserves requirements to limit inflation. Canada, the UK, New Zealand, Australia and Sweden have no reserve requirements and instead use capital requirements.

borrowers will shift downwards and the supply curve of lenders will shift upwards, resulting in smaller transaction volumes.

7 Examples

We consider here few numerical examples to illustrate the concepts and results presented so far. In all examples we will assume for the sake of simplicity, that all banks are homogeneous in all respects except for their initial predictions of their cash positions. Specifically, banks are subjects to the funds' uncertainty shocks generated from the normal CDF with mean 0 and variance $\sigma^2 = 1$. All banks have probability of default $q = 0.001$. Parameters of the Central Bank are: deposit rate $r^d = 0.03$, penalty rate $r^p = 0.05$, the reserve requirements common for all banks are $T = 2.2$.

There will be four banks overall, A, B, C and D, but we start with situations when only some of these banks are present. Banks A and C are identical and relatively good in terms of projections of their reserves, for these banks $s_A = s_C = 3$. With $T = 2.2$ it means that the banks will not meet the reserve requirements with probability $F(2.2 - 3) \approx 0.21$. Their profit for a given parameters in the absence of interbank market is approximately 0.022 according to (2). Banks B and D expect to have fewer reserves, $s_B = 1$ and $s_D = 0.9$. Their probabilities of not meeting the requirements are, respectively, $F(2.2 - 1) \approx 0.88$ and $F(2.2 - 1.1) \approx 0.82$, whereas their profits in the absence of interbank market are approximately -0.061 and -0.066 , respectively. We give these benchmark values for profit in the first row of Table 1. In the same table, for the examples discussed below, when specific banks are present, we identify all the networks consistent with pairwise stable equilibria and report the corresponding loans. We also find the competitive pairwise stable equilibria and report the interest rates and the corresponding banks profits (and below, in parentheses, the profit difference with respect to the empty network). Finally, for the equilibrium networks we compute the total volume of trade in the interbank market (4th column) and the expected loss of funds for the interbank market due to exogenous default (the last column) defined as

$$\text{ELoss} = \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{N}} \ell_{ik} q_k.$$

Two banks. Let us consider the case with only two banks A and B. In this case we can expect that bank A may find profitable to lend some funds to bank B but not *vice versa*. Fig. 1 illustrates the feasibility regions, contract curve and competitive pairwise stable equilibrium for these two banks with bank A as a lender and bank B as a borrower. In this

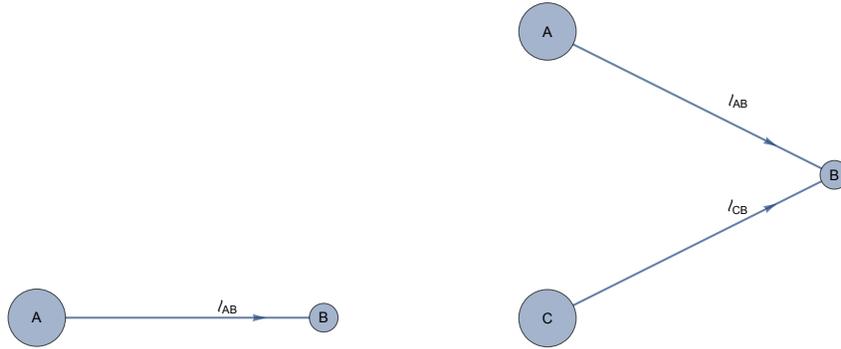


Figure 2: Networks of loans that can be observed in the pairwise stable equilibria in the examples of Section 7 with two and three banks.

case, there is a unique loan amount ℓ_{AB}^* in the pairwise stable equilibrium and a non-empty range of possible interest rates $[r_{AB}^l, r_{AB}^u]$. When the banks are behaving competitively, there is a unique interest rate $r_{AB}^* \in [r_{AB}^l, r_{AB}^u]$. The values of variables⁹ ℓ_{AB}^* and r_{AB}^* can be found in the second row of Table 1. There are no any other pairwise stable equilibrium. Indeed, the empty network is not an equilibrium, as follows from Corollary 1 (and illustration in Fig. 1). Adding the link from B to A to the existing configuration cannot lead to the equilibrium (Lemma 3). Finally, there is no pairwise stable equilibrium with only one loan from B to A, as it follows from Proposition 1 (see also footnote 7).

Three banks. Let us add bank C which is the same as A, $s_C = 3$. It turns out that now the configuration of the previous equilibrium (i.e., with the only link from A to B) is not an equilibrium. This is because C, being not connected but with relatively large reserves, has a low expected marginal rate W_C and can find a profitable deal with bank B. Numerical computations shows that there is only one pairwise stable equilibrium in this case. In this equilibrium banks A and C both lend to B (see the right panel in Fig. 2 and the third row of Table 1). In comparison with the case of two banks, bank B has a higher profit, but bank A has a lower profit, even if it is able now to keep higher reserves. In a sense, even with competitive behavior, an existence of bank C undermines possibility of A to lend at a higher interest rate and get a higher expected profit from the interbank market.

⁹For this example with two banks, we can also compute $r_{AB}^l \approx 0.039$, $r_{AB}^u \approx 0.045$. However, with more banks reporting the interval of interest rates consistent with the bilaterally stable equilibrium is virtually impossible. Indeed, the interval along any existing link depends on the loan amount *and* interest rates on all other existing links.

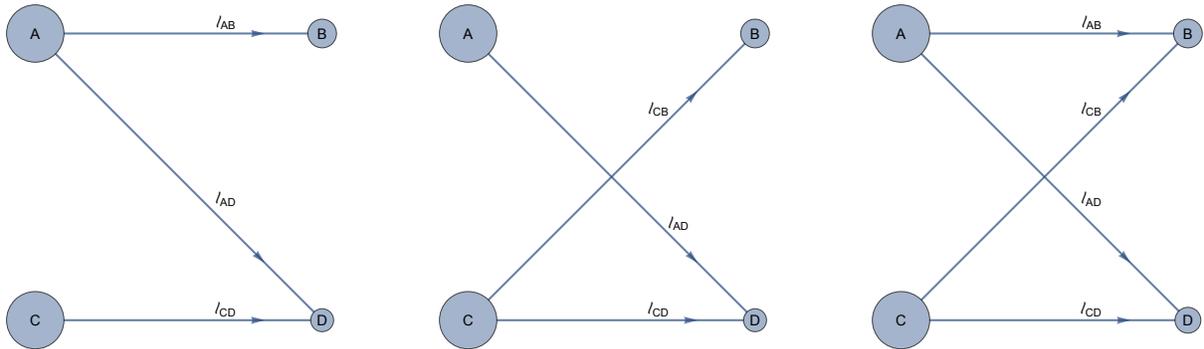


Figure 3: Networks of loans that can be observed in the pairwise stable equilibria in the examples of Section 7 with four banks.

Four banks. We now add bank D which is similar to B, in the sense that it is also a bank with relatively low reserves, though it expects even lower reserves than B, $s_D = 0.9$. The equilibrium configuration of the previous case (A and C lend to B) is not a pairwise stable equilibrium any longer because every of the existing banks and bank D could make a profitable link. The configuration when A and C lend to D is not an equilibrium for the same reason. In this configuration even bank D is willing to lend money to B, even if *ex ante* bank D expects to have less reserves.

Numerically we find that there are three network configurations consistent with pairwise stable equilibrium: (i) A and C lend to D and A lends to B, (ii) A and C lend to D and C lends to B, and (iii) both A and C lend to B and D (see Fig. 3 and the last three rows of Table 1). However, all these equilibria are payoff equivalent for all banks.

8 Data

In this Section we relate our theoretical findings to the data from an overnight interbank market. In particular, we analyse the Electronic Market for Interbank Deposit, e-MID, based in Milan. This is one of the dominant interbank unsecured deposit market on an electronic platform for the Euro area.¹⁰ The e-MID is open to all European banks and European branches of non-European banks. We have access to the e-MID transaction data for the period January 1999 – September 2009. During this period, 350 banks participated in this electronic market. The most active segment of the market consists of the overnight lending.

¹⁰Based on “Euro Money Market Study 2006” by ECB, e-MID accounts for 17% of turnover in unsecured money market in the Euro Area. A more recent 2010 study indicates reduction in the total turnover to 10%.

Example	Configuration	Loan	Volume	Interim Reserves			Interest rate	Profit			Exp Loss	
				C_A	C_B	C_C		C_D	π_A	π_B		π_C
	empty network	0	0	3	1	3	0.9	0.0216	-0.0611	0.0216	-0.0659	0
two banks A and B	A lends to B	$\ell_{AB} = 0.933$	0.933	2.067	1.933			0.0249 (+0.0033)	-0.0582 (+0.0029)			0.00093
three banks A, B and C	A lends to B C lends to B	$\ell_{AB} = 0.623$ $\ell_{CB} = 0.623$	1.2459	2.377	2.2458	2.377		0.0230 (+0.0014)	-0.0555 (+0.0056)	0.0230 (+0.0014)		0.0012
four banks - 1 A, B, C and D	A lends to B A lends to D C lends to D	$\ell_{AB} = 0.9080$ $\ell_{AD} = 0.0500$ $\ell_{CD} = 0.9580$	1.9159	2.0420	1.9080	2.0420	1.9080	0.0251 (+0.0035)	-0.0584 (+0.0027)	0.0251 (+0.0035)		0.0019
four banks - 2 A, B, C and D	A lends to D C lends to B C lends to D	$\ell_{AD} = 0.9580$ $\ell_{CB} = 0.9080$ $\ell_{CD} = 0.0500$	1.9159	2.0420	1.9080	2.0420	1.9080	0.0251 (+0.0035)	-0.0584 (+0.0027)	0.0251 (+0.0035)		0.0019
four banks - 3 A, B, C and D	A lends to B C lends to B A lends to D C lends to D	$\ell_{AB} = 0.6066$ $\ell_{CB} = 0.3014$ $\ell_{AD} = 0.3514$ $\ell_{CD} = 0.6566$	1.9159	2.0420	1.9080	2.0420	1.9080	0.0251 (+0.0035)	-0.0584 (+0.0027)	0.0251 (+0.0035)		0.0019

Table 1: Network configurations in the pairwise stable equilibria and in the competitive pairwise stable equilibria for examples with two, three and four banks.

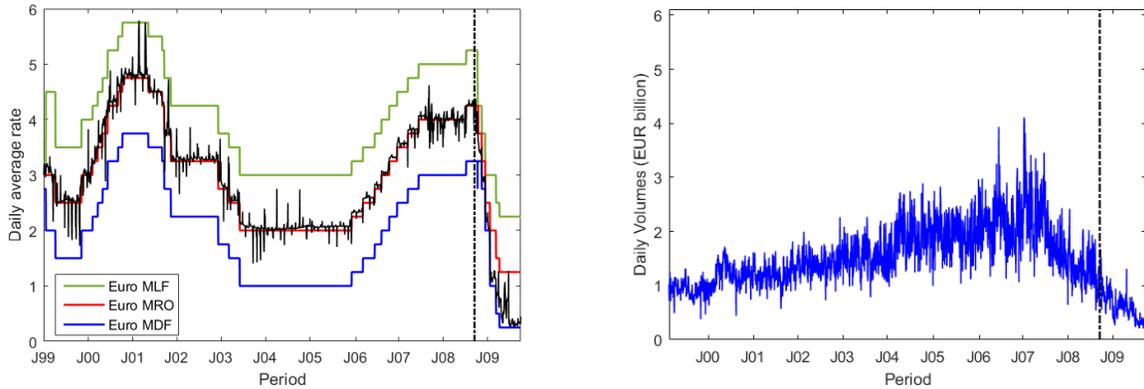


Figure 4: *Left panel:* Daily volume-weighted average e-MID rates (black line) vs the key ECB rates: Euro Marginal Deposit Facility (MLF), Euro Marginal Lending Facility and (MLF) and Euro Main Refinancing Operation (MRO) rates. *Right panel:* daily trading volumes. Dotted black line indicates Lehman Brothers default date (15 September 2008). The labels on the x-axis correspond to the first trading day of January for a given year.

The market is organised in the form of an electronic book, where offers can be submitted and stored. Any bank may post the offer specifying the bid (for borrowing funds) or ask (for lending funds) side, maturity term, rate and amount. Typically the identity of such bank, *the quoter*, is revealed. Best quotes, i.e., the highest bid and the lowest ask, are displayed at the top of the book, with other bids displayed in descending order and other asks displayed in ascending order in the respective side of the book. Any bank may be *an aggressor*, which means that it may pick a quote from the book, negotiate some terms of the offer with the quoter and accept it. The quoter has an option to reject the aggressor's terms after negotiation and after revealing the aggressor's identity. Once the transaction is executed and the book is updated, the transaction terms (maturity, rate, volume) and time stamp are revealed to all market participants.

There is a number of empirical studies using e-MID data. Iori et al. (2008) finds that on daily frequency preferential transactions are limited in the sense that degree distribution does not exhibit fat tails. Moreover, there are virtually no intermediaries. These findings are consistent with our theoretical model. However, for longer aggregation periods (e.g., on monthly, quarterly and annual frequencies) some banks in the e-MID trading network exhibit much higher concentration of links than others, leading to the fat-tails in the degree distribution. Moreover, core-periphery type structures emerge (Finger et al., 2013; Fricke and Lux, 2014). Iori et al. (2015) suggest a trading model with memory that supports preferential attachment and leads to the fat-tailed distribution.

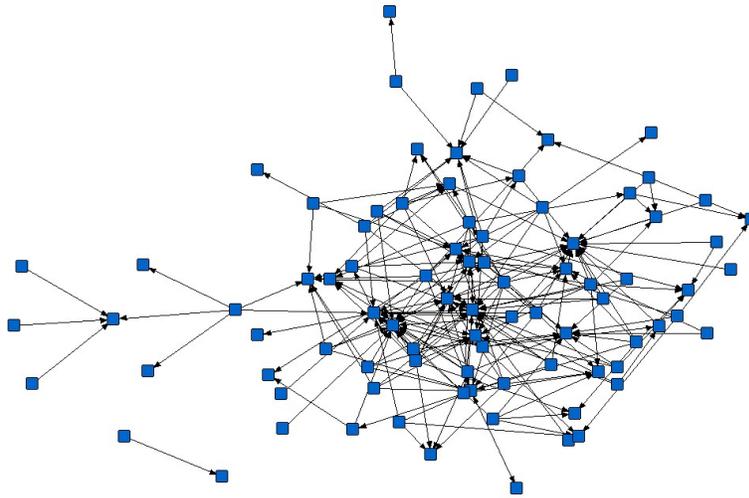


Figure 5: Network of lending and borrowing on the e-MID market. Aggregate transactions for one day. Arrows indicate direction of the loan

As our network formation model is best suited to describe day patterns we will focus on the daily frequency analysis. The left panel of Fig. 4 shows daily volume-weighted average rates on e-MID overnight loans (black line with spikes) along with the key ECB rates. The upper line is the Euro Marginal Lending Facility, the penalty rate at which banks can borrow from the ECB overnight. The lower line is the Euro Marginal Deposit Facility rate at which banks deposit excess reserves with the ECB overnight. Finally, there is a line in between (shown in red) that is the Euro Main Refinancing Operations (MRO) rate at which banks may borrow for one week by providing acceptable collateral to the ECB.

Consistently with the results of our model, we observe that the e-MID rates are between the ECB deposit and lending (penalty) rate, and on average are close to the ECB MRO rate. There is some volatility observed in the average daily rate. Similarly the daily trading volumes (see the right panel of Fig. 4) exhibit substantial variation. In part volatility in the rate and volumes is due to the fact that banks in Euro area are required to meet a minimum reserve requirements on a certain day in the end of the maintenance period. The vertical dotted line indicates 15 September 2008, the date of default of Lehman Brothers. The crisis period after this episode is characterised by small trading volumes and excess reserves as the ECB was providing extra liquidity in the system. Our model predicts that excess liquidity in the system will drive the interest rates down to the Central bank deposit rate, which is consistent with observations in this period.

Fig. 5 illustrates the trading network for a typical trading day (14 January 2009). Arrows

Year	Banks	$\text{Var}(r_{\text{len}})$	$\text{Var}(r_{\text{bor}})$	Prop 2 loops	Prop 3 loops
1999	215	0.0030	0.0027	0.0140	0.0124
2000	196	0.0034	0.0025	0.0105	0.0079
2001	185	0.0037	0.0028	0.0163	0.0087
2002	177	0.0032	0.0024	0.0205	0.0077
2003	179	0.0031	0.0023	0.0174	0.0095
2004	180	0.0029	0.0022	0.0145	0.0064
2005	176	0.0029	0.0021	0.0175	0.0083
2006	177	0.0027	0.0020	0.0133	0.0074
2007	178	0.0029	0.0021	0.0095	0.0035
2008	173	0.0034	0.0024	0.0058	0.0016
2009	153	0.0035	0.0024	0.0043	0.0000
Overall	350	0.0032	0.0024	0.0131	0.0067

Table 2: Characteristics of the e-MID market. For each year column 2 shows the number of active banks on the market; columns 3 and 4 report the volume-weighted average variance of daily individual lending and borrowing rate, respectively; columns 5 and 6 report the proportion of banks involved in 2-loop cycle or 3-loop cycle to the total number of active banks.

indicate the direction of funds flow (from a lender to a borrower). As our model suggests we observe the network with a small number of intermediaries. Remarkably most of the active banks are connected in one giant components. Only two banks are disconnected from the rest of the network. Existence of one giant component including nearly all active banks is a typical daily pattern consistently observed in the data over all considered years. A similar pattern is also predicted by our theoretical model where lenders/borrowers tend to transact with multiple counter-parties.

Our theoretical model predicted that in the competitive pairwise stable equilibrium any borrowing bank will borrow at the same rate from multiple lenders, while lending banks may differentiate the interest rates between different borrowers. To verify whether this holds in the e-MID market we proceed as follows. We identify lenders and borrowers who transacted with more than one counter-party during one day. Then, we calculate for each such bank, the variance of their lending and borrowing rates, respectively. We take a volume-weighted average of these variances over all identified lenders and borrowers and over the number of days for each year. The results are reported in Table 2, columns 3 and 4. The weighted average variance of the identified lenders rates is higher than the weighted average variance of the identified borrowers rate. This result is consistently for all the considered years. Our model predicts that the borrowers' variance should be equal to zero in a short period, but

in practice this is not holding exactly as the borrowers liquidity needs may change over the day and this will trigger changes in demand and, hence, different borrowing rates. Columns 5 and 6 of Table 2 report the proportion of banks involved in two and three cycles. Our model rules out cycles (even in the absence of competitive behavior) and the data confirm that the proportion of banks involved in cycles is negligible. Again, the cycles may arise due to liquidity changes, but not because of strategic considerations.

9 Conclusions

In this paper we built a model of endogenous network formation in the interbank market, with lending banks facing the trade-off between an uncertain gain on the loaned funds and a higher possibility of not meeting reserve requirements. We find that the concept of pairwise stability is sufficient to identify certain network structures which are reminiscent to those that emerge on daily frequency in the real market for interbank lending. Namely, our model predicts the emergence of bipartite network, where lenders and borrowers are connected in one giant component. There are no cycles and intermediaries in the network.

Our model produces intuitive responses to changes in the monetary policy instruments and varying market condition. In particular the model predicts that an increase in the Central Bank lending or borrowing rate or the minimum reserves requirements will increase the average overnight market rate. Larger provability of defaults will increase the average market rate, reduce the volumes and may lead to the “market freeze” where no trade is feasible. The increase in the uncertainty about daily reserves will lower the transition volumes.

In the further analysis, we may consider a stricter version of network equilibrium configurations than the stability concept we employed here. For instance, banks may start to reconsider several links at the same time and they also can suggest doing the same to their counter-parties. This refinement is similar to the concept of bilateral stability studied in a different context by Goyal and Vega-Redondo (2007). Furthermore, banks may not behave myopically, but instead foresee which further changes to the network structure will follow their particular decision to establish or sever a link, and in this case one could for example adapt the framework of Herings et al. (2009).

There are many other possible directions in which the model can be generalized to be more realistic. First, one can relax an assumption of risk neutrality of banks (i.e., expected profit maximization), which rules out any incentive of banks for diversification in their lending behavior. When banks become sensitive to risk, lenders who face a counter-party default risk will be more willing to diversify their supply of lending funds through various banks.

In such a model, it would be interesting to study the case when probabilities of default of different banks are not independent. Second, one can introduce further effect to the interbank market by letting default probabilities be affected by the decision in the interbank market. This would not change the borrower's incentives but would make lenders' tasks more complicated introducing nonlinearities in their behavior.

References

- Acemoglu D., Ozdaglar A., Tahbaz-Salehi A., 2015. Systemic risk and stability in financial networks. *American Economic Review* 105, 564–608.
- Afonso G., Lagos R., 2015. Trade dynamics in the market for federal funds. *Econometrica* 83, 263–313.
- Allen F., Gale D., 2000. Financial contagion. *Journal of Political Economy* 108, 1–33.
- Babus A., Hu T.W., 2015. Endogenous intermediation in over-the-counter markets. Technical report, CEPR discussion paper.
- Bech M., Monnet C., 2013. The impact of unconventional monetary policy on the overnight interbank market. In: *Liquidity and Funding Markets*, RBA Annual Conference Volume. Reserve Bank of Australia.
- Blasques F., Bräuning F., Van Lelyveld I., 2015. A dynamic network model of the unsecured interbank lending market. Technical Report BIS Working Paper, 491, Bank for International Settlements.
- Blume L.E., Easley D., Kleinberg J., Tardos E., 2009. Trading networks with price-setting agents. *Games and Economic Behavior* 67, 36–50.
- Cabrales A., Gottardi P., Vega-Redondo F., 2014. Risk-sharing and contagion in networks. Technical Report 4715, Center for Economic Studies and Ifo Institute.
- Corominas-Bosch M., 2004. Bargaining in a network of buyers and sellers. *Journal of Economic Theory* 115, 35–77.
- Craig B., Von Peter G., 2014. Interbank tiering and money center banks. *Journal of Financial Intermediation* 23, 322–347.

- Elliott M., Golub B., Jackson M.O., 2014. Financial networks and contagion. *American Economic Review* 104, 3115–53.
- Finger K., Fricke D., Lux T., 2013. Network analysis of the e-mid overnight money market: the informational value of different aggregation levels for intrinsic dynamic processes. *Computational Management Science* 10, 187–211.
- Fricke D., Lux T., 2014. Core–periphery structure in the overnight money market: evidence from the e-mid trading platform. *Computational Economics* 45, 1–37.
- Gai P., Kapadia S., 2010a. Contagion in financial networks. *Proceedings of the Royal Society of London A* 466, 439. URL <http://rspa.royalsocietypublishing.org/content/early/2010/03/18/rspa.2009.0410>.
- Gai P., Kapadia S., 2010b. Liquidity hoarding, network externalities, and interbank market collapse. Technical report.
- Glasserman P., Young H.P., 2015. How likely is contagion in financial networks? *Journal of Banking and Finance* 50, 383–399.
- Goyal S., Vega-Redondo F., 2007. Structural holes in social networks. *Journal of Economic Theory* 137, 460–492.
- Haldane A.G., May R.M., 2011. Systemic risk in banking ecosystems. *Nature* 469, 351–355.
- Herings P.J.J., Mauleon A., Vannetelbosch V., 2009. Farsightedly stable networks. *Games and Economic Behavior* 67, 526–541.
- Hommes C., van der Leij M., in 't Veld D., 2014. The formation of a core periphery structure in heterogeneous financial networks. Technical Report 14-04, CeNDEF, University of Amsterdam.
- Iori G., De Masi G., Precup O.V., Gabbi G., Caldarelli G., 2008. A network analysis of the italian overnight money market. *Journal of Economic Dynamics and Control* 32, 259–278.
- Iori G., Mantegna R.N., Marotta L., Micciche S., Porter J., Tumminello M., 2015. Networked relationships in the e-mid interbank market: A trading model with memory. *Journal of Economic Dynamics and Control* 50, 98–116.
- Jackson M.O., Wolinsky A., 1996. A strategic model of social and economic networks. *Journal of Economic Theory* 71, 44–74.

- Kranton R., Minehart D.F., 2001. A theory of buyer-seller networks. *American Economic Review* 91, 485–508.
- van Lelyveld I., in 't Veld D., 2014. Finding the core: Network structure in interbank markets. *Journal of Banking and Finance* 49, 27–40.
- Manea M., 2011. Bargaining in stationary networks. *American Economic Review* 101, 2042–2080.
- Nava F., 2015. Efficiency in decentralized oligopolistic markets. *Journal of Economic Theory* 157.
- Poole W., 1968. Commercial bank reserve management in a stochastic model: implications for monetary policy. *The Journal of Finance* 23, 769–791.

APPENDIX

A Derivatives, First and Second order conditions

We compute the derivatives of π_i given in (4) with respect to the borrowing quantity ℓ_{ji} from bank j and also with respect to the lending quantity ℓ_{ij} to bank j .

Derivatives for the borrower. When bank i wants to borrow from bank j , its marginal profit is

$$\begin{aligned} \frac{\partial \pi_i}{\partial \ell_{ji}} &= -r_{ji} + r^d + F_i(T_i - C_i)(r^p - r^d) - (C_i - T_i)f_i(T_i - C_i)(r^p - r^d) + \\ &\quad + (C_i - T_i)f_i(T_i - C_i)(r^p - r^d) = \\ &= -r_{ji} + r^d + (r^p - r^d) F_i\left(T_i - s_i + \sum_k \ell_{ik} - \sum_m \ell_{mi}\right) \end{aligned} \quad (13)$$

Setting this to zero we get the first-order condition for the borrower (12).

The second derivative reads

$$\frac{\partial^2 \pi_i}{\partial \ell_{ji}^2} = -(r^p - r^d)f_i(T_i - C_i) < 0.$$

The derivative of profit by the interest rate is

$$\frac{\partial \pi_i}{\partial r_{ji}} = -\ell_{ji}.$$

When we impose that bank i lends money to bank j then the computations above imply that the slope of indifference curve of borrower j in the coordinates (ℓ_{ij}, r_{ij}) is given by

$$-\frac{\partial \pi_j / \partial \ell_{ij}}{\partial r_{ij}} = \frac{-r_{ij} + r^d + (r^p - r^d) F_j(T_j - C_j)}{\ell_{ij}}.$$

This result is then substituted to the right hand-side of condition (9) in order to get the result (10) in the main text.

Derivatives for the lender. When bank i wants to lend to bank j , its marginal profit is

$$\begin{aligned} \frac{\partial \pi_i}{\partial \ell_{ij}} &= r_{ij}(1 - q_j) - q_j - r^d - F_i(T_i - C_i)(r^p - r^d) + (C_i - T_i)f_i(T_i - C_i)(r^p - r^d) - \\ &\quad - (C_i - T_i)f_i(T_i - C_i)(r^p - r^d) = \\ &= r_{ij}(1 - q_j) - q_j - r^d - (r^p - r^d) F_i \left(T_i - s_i + \sum_k \ell_{ik} - \sum_m \ell_{mi} \right) \end{aligned} \quad (14)$$

Setting this to zero we get the first-order condition for the lender (11). The second-order derivative is given by

$$\frac{\partial^2 \pi_i}{\partial \ell_{ij}^2} = -(r^p - r^d) f_i(T_i - C_i) < 0$$

which is the same as for the borrower.

The derivative of profit by the interest rate is

$$\frac{\partial \pi_i}{\partial r_{ij}} = \ell_{ij}(1 - q_j).$$

When we impose that bank i lends money to bank j then the computations above imply that the slope of indifference curve of lender i in the coordinates (ℓ_{ij}, r_{ij}) is given by

$$-\frac{\partial \pi_i / \partial \ell_{ij}}{\partial \pi_i / \partial r_{ij}} = -\frac{r_{ij}(1 - q_j) - q_j - r^d - (r^p - r^d) F_i(T_i - C_i)}{\ell_{ij}(1 - q_j)}.$$

This result is then substituted to the left hand-side of condition (9) in order to get the result (10) in the main text.